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Stability of Spatial Equilibrium

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Stability of Spatial Equilibrium*

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Abstract

Dynamical stability of equilibrium is often difficult to know when the number of variables exceeds 4. This is because all the eigenvalues of the Jacobian matrix are not analytically solvable. However, we obtained stability conditions for a general class of migration dynamics without computing eigenvalues. We showed that a spatial equilibrium is stable in the presence of strong congestion diseconomies, but unstable in the presence of strong agglomeration economies. We also showed in the case of negligible interregional externalities that stability is not affected by the city location and the equilibrium distribution and that a stable equilibrium generically exists.

Keywords: asymptotic stability, existence of stable equilibrium, positive definite dynamics, economic geography, interregional externality.

J.E.L. Classification: C62, C73, R23.

1 Introduction

Ever since the work of Krugman (1991), there has been a growing literature on interregional general equilibrium analysis in economic geography (Fujita and Thisse, 2002). In the literature, all economic variables constituting utility functions are determined by the spatial distribution of population as reduced forms in the short-run equilibrium, and then migration takes according to interregional utility differentials, resulting in the long-run spatial equilibrium such that no individual has an incentive to migrate.

It is definitely important to know if there exists a spatial equilibrium, if this equilibrium is stable against any perturbations, and if there exists a stable equilibrium. Due to the complexity of the model structure, most of the literature on economic geography focuses only on

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the case of two regions. However, Fujita, Krugman and Venables (1999), Brakman, Garretsen and Marrewijk (2001) and Tabuchi, Thisse and Zeng (2002) among others, attempt to analyze the economic geography of more than 2 regions, in spite of the fact that stability of a spatial equilibrium and its existence in multiregional systems are hardly clarified so far in the literature.

It is well-known that the asymptotic stability is determined by the signs of real parts of eigenvalues of Jacobian of a dynamical system. Problems arise in the case of the large number of regions. When the number is greater than 4, the characteristic equation cannot be solved analytically, and hence all eigenvalues may not be computed even numerically. Attempts have been made by Miyao (1978) and Ginsburgh, Papageorgiou and Thisse (1985), Tabuchi (1986) and Zeng (2002) to obtain conditions for stability of specific dynamics. However, the conditions seem to be far from necessary and sufficient.

The primary purpose of this paper is to derive more useful and general stability conditions for a class of multiregional dynamics including the replicator dynamic (Taylor and Jonker, 1978), without computations of high-order polynomials. It is clarified that dynamics are stable (resp. unstable) in the presence of negative (resp. positive) externalities. When interregional externalities are negligible, but intraregional externalities are not, we obtain necessary and sufficient conditions for asymptotic stability.

The secondary purpose is to show the existence of a stable equilibrium. Indeed, the existence of an equilibrium is shown by Ginsburgh *et al.* (1985) for any continuous utilities, but the existence of a *stable* equilibrium has not yet explored in the previous literature. Scarf's (1960) counterexample demonstrates that a stable equilibrium does not necessarily exist in general. However, we show its existence in the case of negligible interregional externalities and differentiable utilities for a class of dynamics including the replicator dynamic.

We would like to emphasize that the model of our paper is general enough in two respects. First, although we deal with regions, they may be interpreted as various concepts – for example, strategies in game theory and subgroups such as clubs in public finance. Second, our class of dynamics is wide. It includes all kinds of migration dynamics, where migration always occurs from low- to high-utility regions, such as the replicator dynamic and gravity models.

The remainder of the paper is organized as follows. First, we define spatial equilibrium and asymptotic stability, and formulate a class of multiregional dynamical system in Sections 2. We then derive the stability conditions of the system in Section 3. By imposing a further assumption of negligible interregional externalities, we obtain necessary and sufficient conditions for stability in Section 4. Finally, we investigate the existence of stable equilibrium in Section5. Section 6 concludes.

2 Spatial equilibrium and dynamics

The space economy is made of $n \geq 2$ regions, where the total population is fixed and normalized to 1. Let $x_i \in [0,1]$ denote the population share in region $i = 1, \ldots, n$ and let

$$X \equiv \left\{ \boldsymbol{x} = (x_1, \ldots, x_n)', \quad \sum_{i=1}^{n} x_i = 1 \quad \text{and} \quad x_i \geq 0 \right\}.$$

Suppose the utility in each region is expressed as a function of population distribution.[1] Let $u_i(\boldsymbol{x})$ be the (indirect) utility level residing in region $i = 1, \ldots, n$. Assuming a zero cost of migration, each household freely migrates between regions so as to maximize its utility, resulting in an equal utility level u^*. The distribution $\boldsymbol{x}^*$ is a *spatial equilibrium* when no individual can receive a higher utility level by migrating to another region. Formally, a distribution $\boldsymbol{x}^* \in X$ is an equilibrium if u^* exists such that

$$u_i(\boldsymbol{x}^*) = u^* \quad \text{if} \quad x_i^* > 0,$$
$$u_i(\boldsymbol{x}^*) \leq u^* \quad \text{if} \quad x_i^* = 0.$$

The equality means that the equilibrium utility is constant across regions with positive population. The inequality implies that some regions may have zero population in equilibrium with lower utility levels than other regions. We suppose that $u_i(\boldsymbol{x})$ is continuous so that a spatial equilibrium $\boldsymbol{x}^*$ always exists from Proposition 1 of Ginsburgh *et al.* (1985). We further suppose that $u_i(\boldsymbol{x})$ is twice differentiable around $\boldsymbol{x}^*$.

Now let $\boldsymbol{x}^* = (x_1^*, \ldots, x_n^*)'$ be an interior equilibrium.[2] In order to seek local stability conditions, we limit our concern to a neighborhood of $\boldsymbol{x}^*$. Following well-tradition of migration theory (Greenwood, 1975), we assume that individuals rationally migrate from lower- to higher-utility regions. To be more specific, the population (share) migrating from the origin region j to the destination region $i (\neq j)$ is proportional to the utility differential

$$\dot{x}_{ji} = f_{ij}(\boldsymbol{x})[u_i(\boldsymbol{x}) - u_j(\boldsymbol{x})] \qquad \text{for } i, j \in \{1, \ldots, n\}, \tag{1}$$

where $f_{ij}(\boldsymbol{x}) > 0$ is interpreted as the adjustment speed of migration, distance between regions, or "gravitational effects" between regions.

Since $\dot{x}_{ij} = -\dot{x}_{ji}$, symmetry $f_{ij}(\boldsymbol{x}) = f_{ji}(\boldsymbol{x})$ holds for all $i, j = 1, \ldots, n$. Summing (1) over j yields the dynamical system of multi-regions as:

$$\dot{\boldsymbol{x}} = F(\boldsymbol{x}) \cdot U(\boldsymbol{x}), \tag{2}$$

[1] Such a reduced form of the indirect utility is obtained as a reduced form of maximizing behavior by firms and households assuming that all prices and quantities are uniquely determined. Examples are found in the literature on economic geography such as Fujita *et al.* (1999), Tabuchi *et al.* (2002) and Forslid and Ottaviano (2002) among others.

[2] If the equilibrium hits the corner, then the same analysis applies by reducing the number of regions.

where

$$F(\boldsymbol{x}) \equiv \begin{pmatrix} f_1(\boldsymbol{x}) & -f_{12}(\boldsymbol{x}) & \cdots & -f_{1n}(\boldsymbol{x}) \\ -f_{12}(\boldsymbol{x}) & f_2(\boldsymbol{x}) & \cdots & -f_{2n}(\boldsymbol{x}) \\ \vdots & \vdots & \ddots & \vdots \\ -f_{1n}(\boldsymbol{x}) & -f_{2n}(\boldsymbol{x}) & \cdots & f_n(\boldsymbol{x}) \end{pmatrix}, \qquad U(\boldsymbol{x}) \equiv \begin{pmatrix} u_1(\boldsymbol{x}) \\ u_2(\boldsymbol{x}) \\ \vdots \\ u_n(\boldsymbol{x}) \end{pmatrix},$$

and $f_i(\boldsymbol{x}) \equiv \sum_{j\neq i}^{n} f_{ij}(\boldsymbol{x})$ for $i = 1, \ldots, n$.

Now we expand the RHS of (2) by Taylor's theorem in the neighborhood of $\boldsymbol{x}^*$,

$$\dot{x}_i = \sum_{j\neq i} f_{ij}(\boldsymbol{x})[u_i(\boldsymbol{x}) - u_i(\boldsymbol{x}^*) + u_j(\boldsymbol{x}^*) - u_j(\boldsymbol{x})] = \sum_{j\neq i} f_{ij}(\boldsymbol{x}) \sum_k (u_{ik} - u_{jk})(x_k - x_k^*), \quad (3)$$

where

$$u_{ik} \equiv \left. \frac{\partial u_i(\boldsymbol{x})}{\partial x_k} \right|_{\boldsymbol{x}=\boldsymbol{x}^*}.$$

In matrix representation, (3) can be written as

$$\dot{\boldsymbol{x}} = F(\boldsymbol{x}^*) \cdot \partial U \cdot (\boldsymbol{x} - \boldsymbol{x}^*), \quad (4)$$

where

$$\partial U \equiv \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \cdots & u_{nn} \end{pmatrix}.$$

Dynamic (4) is called a *positive definite (PD) dynamic*[3] if the following conditions additionally hold (Hopkins, 1999).

(i) Every element of F is continuously differentiable in $\boldsymbol{x}$.

(ii) $\boldsymbol{y}'F\boldsymbol{y} > 0$ for all $\boldsymbol{y} \in \mathbb{R}^n$ which is not a multiple of $\mathbf{1} = (1, \ldots, 1)'$.

Note that condition (i) ensures a unique equilibrium for both dynamic (2) and (4), as shown by Hirsch and Smale (1974, p.164). Condition (ii) means F is positive definite in the $n-1$-dimensional space, which is orthogonal to $(1, \ldots, 1)'$.

The class of PD dynamics is large enough. For example, if we set $f_{ij}(\boldsymbol{x}) = \kappa x_i x_j$ for all i, j, then the PD dynamics (2) turn out to be the replicator dynamic, which is used by Fujita *et al.* (1999):

$$\dot{x}_i = \kappa x_i \left[u_i(\boldsymbol{x}) - \sum_j x_j u_j(\boldsymbol{x})\right] \qquad \text{for } i = 1, \ldots, n.$$

[3]Hofbauer and Sigmund (1998) call such a dynamic simply *adaptive dynamic*. This dynamic is *weak compatible* with a fitness function $u_i(\boldsymbol{x})$ (Friedman, 1991).

If we set $f_{ij}(\boldsymbol{x}) = \kappa/n$, then (2) is reduced to the dynamics used by Tabuchi (1986), Friedman (1991) and Zeng (2002):

$$\dot{x}_i = \kappa\,[u_i(\boldsymbol{x}) - \frac{1}{n}\sum_j u_j(\boldsymbol{x})] \qquad \text{for} \;\; i = 1, \ldots, n.$$

It will be shown later that stability conditions of these dynamics are independent of $f_{ij}(\boldsymbol{x})$ when interregional externalities are much smaller than intraregional ones.

A spatial equilibrium $\boldsymbol{x}^*$ is *asymptotically stable* if, for any positive ϵ, there exists a neighborhood $N(\boldsymbol{x}^*)$ of $\boldsymbol{x}^*$ such that for any $\boldsymbol{x}^0 \in N(\boldsymbol{x}^*)$, the solution $\boldsymbol{x}(t) = (x_1(t), \ldots, x_n(t))'$ of a given dynamical system with an initial value $\boldsymbol{x}^0(0) = \boldsymbol{x}^0$ satisfies $||\boldsymbol{x}(t) - \boldsymbol{x}^*||(\equiv \max_{i=1,\ldots,n}|x_i(t) - x_i^*|) < \epsilon$ for any time $t \geq 0$ and $\lim_{t\to\infty}\boldsymbol{x}(t) = \boldsymbol{x}^*$. It is known that equilibrium $\boldsymbol{x}^*$ of (2) and (4) is asymptotically stable if all the real parts of eigenvalues of $F(\boldsymbol{x}^*) \cdot \partial U$ are negative.

3 Interactions among regions

We first consider the case that each utility function depends on the whole population distribution ($u_i(\boldsymbol{x})$), in this section, and then focus on the case that each utility function depends only on its population ($u_i(x_i)$).

The Jacobian matrix of (2) or (4) is $F(\boldsymbol{x}^*)\partial U$, which is denoted by $A = [a_{ij}]_{n\times n}$ and

$$a_{ij} = \begin{cases} \displaystyle\sum_{k=1,k\neq i}^{n} (u_{ii} - u_{ki})f_{ik} & \text{for } i = j, \\ \displaystyle\sum_{k=1,k\neq i}^{n} (u_{ij} - u_{kj})f_{ik} & \text{for } i \neq j. \end{cases}$$

where $f_{ik} = f_{ik}(\boldsymbol{x}^*)$. Since matrix F has positive and weakly dominant diagonals, we shall make use of Geršgorin theorem (Horn and Johnson, 1985) on diagonal dominance in order to examine the signs of the largest and the second-largest eigenvalues, paying attention to the multiplicity of an eigenvalue $x = 0$.

Denote the cofactor of u_{ij} in ∂U by $\mathcal{U}_{ij} \overset{\text{def}}{=} (-1)^{i+j}\left|\widetilde{\partial U}_{ij}\right|$, where $\widetilde{\partial U}_{ij}$ is a submatrix formed from ∂U by deleting the i-th row and the j-th column. From Propositions A.1, A.2 and A.3 in Appendix A, we obtain the following result.

Lemma 1 *A spatial equilibrium* $\boldsymbol{x}^*$ *of (4) is stable if either (i) or (ii) holds.*
(i) For all j, $\min_{i=1,\ldots,n,\ i\neq j}\{a_{ij}\} > 0$ *holds.*
(ii) For one j, $\max_{i=1,\ldots,n,\ i\neq j}\{a_{ij}\} < 0$ *holds, while for all other* j, $\min_{i=1,\ldots,n,\ i\neq j}\{a_{ij}\} \geq 0$ *holds. In addition,* $(-1)^n\sum_{i=1}^n\sum_{j=1}^n \mathcal{U}_{ij} < 0$ *holds.*

On the other hand, $\boldsymbol{x}^*$ *is unstable if either (iii) or (iv) holds.*

(iii) For at least two j, $\max_{i=1,\dots,n,\ i\neq j}\{a_{ij}\} < 0$ holds, and

$$|\partial U| \neq 0, \qquad \sum_{i=1}^{n}\sum_{j=1}^{n}\mathcal{U}_{ij} \neq 0. \tag{5}$$

(iv) For one j, $\max_{i=1,\dots,n,\ i\neq j}\{a_{ij}\} < 0$ holds, while for all other j, $\min_{i=1,\dots,n,\ i\neq j}\{a_{ij}\} \geq 0$ holds. In addition, $(-1)^n\sum_{i=1}^{n}\sum_{j=1}^{n}\mathcal{U}_{ij} > 0$ holds.

From Lemma 1 (i) and (iii), we establish the following.

Theorem 1 *Spatial equilibrium $\boldsymbol{x}^*$ is stable if*

$$u_{ij} > \frac{\sum_k f_{ik}u_{kj}}{\sum_k f_{ik}} \qquad \forall i,j,\ i\neq j, \tag{6}$$

while $\boldsymbol{x}^*$ is unstable if

$$u_{ij} < \frac{\sum_k f_{ik}u_{kj}}{\sum_k f_{ik}} \qquad \forall i \text{ and at least two } j,\ i\neq j. \tag{7}$$

While the LHS of (6) and (7) is the change in u_i due to the change in x_j, the RHS is the change in the average utility (the population-weighted average $\sum_k x_j u_{kj}$ if $f_{ij} = \kappa x_i^* x_j^*$ and the simple average $\sum_k u_{kj}/n$ if $f_{ij} = \kappa/n$). Hence, condition (6) implies that *migration to region j increases utility $i(\neq j)$ relative to the average utility.* That is, any small migration to region j will induce reverse migration from region j to regions $i = 1,\dots,j-1,j+1,\dots,n$. Since $\partial\dot{x}_i/\partial x_j|_{\boldsymbol{x}=\boldsymbol{x}^*} = a_{ij} > 0$ for all $i\neq j$, negative feedbacks prevail.

On the other hand, condition (7) implies that a small migration to region j decreases the utility of all other regions relative to the average utility. In other words, any small migration to region j will induce further migration to the same region j from regions $i = 1,\dots,j-1,j+1,\dots,n$. Since $\partial\dot{x}_j/\partial x_j|_{\boldsymbol{x}=\boldsymbol{x}^*} = a_{jj} = -\sum_{i=1,i\neq j}^{n} a_{ij} > 0$, positive feedback are prevalent. If this happens to occur in 2 or more regions, then the equilibrium cannot be stable. (If this occurs only in 1 region, then the equilibrium may be stable or unstable because constraint $\sum_i x_i = 1$ may prevent the positive feedback. This corresponds to cases (ii) and (iv) in Lemma 1.)

Conditions (6) and (7) can be rewritten, respectively, as

$$u_{jj} < u_{ij} + \sum_{k=1,k\neq i,j}^{n}(u_{ij}-u_{kj})\frac{f_{ik}}{f_{ij}} \qquad \forall i,j,\ i\neq j,$$

$$u_{jj} > u_{ij} + \sum_{k=1,k\neq i,j}^{n}(u_{ij}-u_{kj})\frac{f_{ik}}{f_{ij}} \qquad \forall i \text{ and at least two } j,\ i\neq j.$$

The first inequality implies the PD dynamics (4) is stable if the intraregional interactions u_{jj} are much smaller (larger in absolute values) than the collections of the interregional ones u_{ij}'s ($i\neq j$). That is, *sufficiently large congestion diseconomies within all regions ensure the stability of* *interior equilibrium for any PD dynamics.* The second inequality shows that *large agglomeration economies within two regions are sufficient to destroy the stability of interior equilibrium.*

The advantage of Theorem 1 is as follows. In general, when the order of the characteristic equation exceeds 4, it is not analytically solvable. In this case, one may not be able to find all eigenvalues of the Jacobian matrix numerically. On the other hand, all the eigenvalues are not needed in Lemma 1 and Theorem 1, which require the four basic operations of arithmetic only.

Finally, symmetry in the regional system substantially simplifies the stability conditions. Hofbauer and Sigmund (1998) showed that if $u_{ij} = u_{ji}$ holds for all i and j, the stability condition of the replicator dynamic is given by

$$(-1)^i \begin{vmatrix} 0 & \mathbf{1}' \\ \mathbf{1} & \dfrac{\partial U_i + \partial U_i'}{2} \end{vmatrix} > 0 \qquad \forall i \geq 2. \tag{8}$$

The eigenvalues need not be computed, while (8) is easy to compute as in Theorem 1. Moreover, it is not difficult to extend the result as follows.[4] *If* $u_{ij} + u_{jk} + u_{ki} = u_{ik} + u_{kj} + u_{ji}$ *holds for all* i*,* j *and* k *(which includes* $u_{ij} = u_{ji}$ *for all* i *and* j*), the stability condition of any PD dynamic is given by (8).*

4 Interactions within regions

In spatial economy, the intraregional market interactions are usually much stronger than the interregional market interactions. In this case, we assume

$$u_i(\boldsymbol{x}) = u_i(x_i), \tag{9}$$

which may be justified when the impacts of own population are much stronger than those of other populations, i.e., $|u_{ii}| \gg |u_{ij}|$ for all $i \neq j$. For example, urban costs and benefits such as congestion and product variety are usually closely related to population size within a region only, but not to the populations of other regions.

Without loss of generality, let

$$u_{11} \geq u_{22} \geq \cdots \geq u_{nn}. \tag{10}$$

Then, we have the following nearly necessary and sufficient conditions for the stability of spatial equilibrium.

Proposition 1 *In the case of (9), any PD dynamic (4) is asymptotically stable if (i) or (ii) holds.*

(i) $u_{11} \leq 0$ *and* $u_{22} < 0$,

[4] Proofs may be obtained from the authors upon request.

(ii) $u_{11} > 0 > u_{22}$ *and* $\sum_i 1/u_{ii} > 0$.

Any PD dynamic (4) is asymptotically unstable if (iii) or (iv) holds.

(iii) $u_{11} > 0 > u_{22}$ *and* $\sum_i 1/u_{ii} < 0$,

(iv) $u_{11} > 0$ *and* $u_{22} \geq 0$.

The proof is contained in the Appendix B. Surprisingly, Proposition 1 says that stability conditions are determined only by the marginal changes in utilities u_{ii}, but are independent of $f_{ij}(\boldsymbol{x}^*)$. That is, unlike Theorem 1, *Proposition 1 holds for any geographical location of cities, for any speed of adjustments by interregional migration, and for any equilibrium distribution of population in the case of negligible interregional externalities.* For example, observe that the following dynamics for $n = 3$ are very different

$$\begin{cases} \dot{x}_1 &= x_1\left[(1-x_1)u_1(x_1) - x_2u_2(x_2) - x_3u_3(x_3)\right] \\ \dot{x}_2 &= x_2\left[(1-x_2)u_2(x_2) - x_3u_3(x_3) - x_1u_1(x_1)\right] \end{cases}$$

$$\begin{cases} \dot{x}_1 &= 1000u_1(x_1) - u_2(x_2) - 999u_3(x_3) \\ \dot{x}_2 &= 2u_2(x_2) - u_1(x_1) - u_3(x_3) \end{cases}$$

where the first dynamic is the replicator with $f_{ij} = x_ix_j$ and the second is with $f_{13} = 999$ (regions 1 and 3 are very close). Nevertheless, their stability conditions of the common equilibrium are the same and given by Proposition 1.

Stability condition (i) in Proposition 1 says that if an increasing population lowers its utility level in each region, no individual has an incentive to migrate, and hence it is stable. This implies that the prevalence of congestion diseconomies in all regions should ensure the stability of interior equilibrium. On the other hand, instability condition (iv) states that agglomeration economies only in two regions are sufficient to break the stability of the spatial equilibrium, which is consistent with the previous section.

If there is only one positive slope of the utility function, conditions (ii) and (iii) apply. For $n = 2$, suppose a small (net) migration were to occur from region 2 with $u_{22} < 0$ to region 1 with $u_{11} > 0$, then the utility levels in both regions would increase. In case (ii), since

$$\frac{\Delta x_1}{\Delta u_1} + \frac{\Delta x_2}{\Delta u_2} = \frac{\Delta x_1}{\Delta u_1} - \frac{\Delta x_1}{\Delta u_2} > 0, \tag{11}$$

region 2's utility increases more than region 1's. This necessarily generates the reverse migration from regions 1 to 2, which restores the original equilibrium $\boldsymbol{x}^*$. In case (iii), however, region 2's utility increases less, leading to instability.

A similar principle holds for $n \geq 3$. Suppose a small migration from regions $i = 2, \ldots, n$ with $u_{ii} < 0$ to region 1 with $u_{11} > 0$ takes place. The change in the utility level in each region is illustrated in Figure 1. Since $\Delta x_1 = -\sum_{i=2}^n \Delta x_i > 0$, the last stability condition in case (ii) is rewritten as

$$\frac{\Delta x_1}{\Delta u_1} + \sum_{i=2}^{n} \frac{\Delta x_i}{\Delta u_i} = \sum_{i=2}^{n} \Delta x_i \left(\frac{1}{\Delta u_i} - \frac{1}{\Delta u_1}\right) > 0. \tag{12}$$

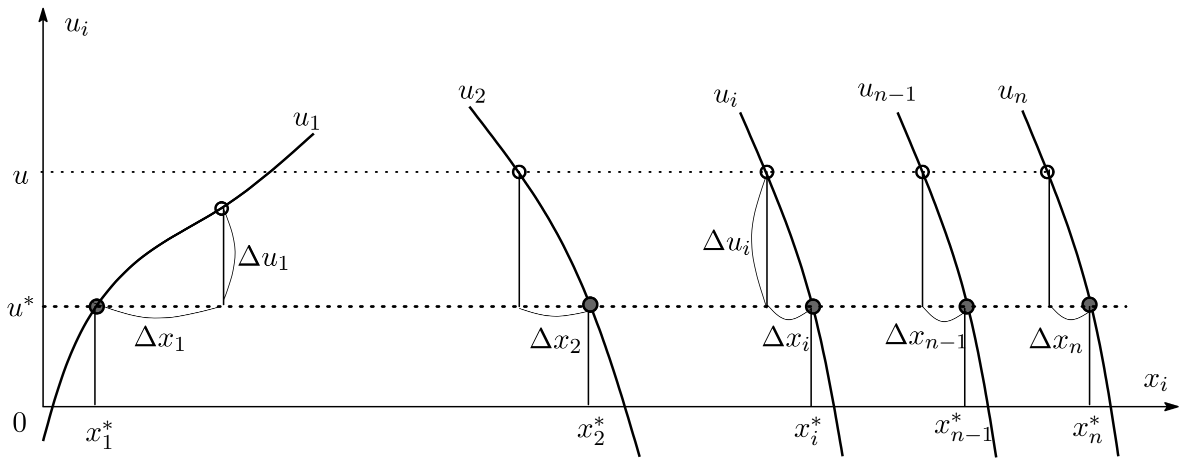

Figure 1: Stability in a multiregional system

If each Δu_i is the same for all $i = 2, \ldots, n$ as in Figure 1, then (12) is reduced to (11). Since the utility increase in region 1 is smaller than that in other regions ($\Delta u_1 < \Delta u_i$), the stability of the multiregional system should be guaranteed.

5 Existence of stable equilibrium

Another important issue is existence of stable equilibrium, which can be shown below in the case of intraregional interactions (9). In the case of interregional interactions stable equilibria do not necessarily exist even in PD dynamics as exemplified by Scarf (1960). Scarf's (1960) equation (12) with $a = 2$ and $b = 4$ is:

$$\begin{cases} \dot{x}_1 &= 2u_1(\boldsymbol{x}^*) - u_2(\boldsymbol{x}^*) - u_3(\boldsymbol{x}^*) \\ \dot{x}_2 &= 2u_2(\boldsymbol{x}^*) - u_1(\boldsymbol{x}^*) - u_3(\boldsymbol{x}^*), \end{cases}$$

where

$$u_1(\boldsymbol{x}^*) = \frac{8x_1^{\frac{2}{3}}}{3(4x_1^{\frac{2}{3}} + x_2^{\frac{2}{3}})} + \frac{2x_3}{3x_1^{\frac{1}{3}}(x_1^{\frac{2}{3}} + 4x_3^{\frac{2}{3}})} + \frac{4x_2^{\frac{2}{3}}}{3(4x_2^{\frac{2}{3}} + x_3^{\frac{2}{3}})} + \frac{x_1}{3x_2^{\frac{1}{3}}(4x_1^{\frac{2}{3}} + x_2^{\frac{2}{3}})}$$

$$u_2(\boldsymbol{x}^*) = \frac{4x_1^{\frac{2}{3}}}{3(4x_1^{\frac{2}{3}} + x_2^{\frac{2}{3}})} + \frac{x_3}{3x_1^{\frac{1}{3}}(x_1^{\frac{2}{3}} + 4x_3^{\frac{2}{3}})} + \frac{8x_2^{\frac{2}{3}}}{3(4x_2^{\frac{2}{3}} + x_3^{\frac{2}{3}})} + \frac{2x_1}{3x_2^{\frac{1}{3}}(4x_1^{\frac{2}{3}} + x_2^{\frac{2}{3}})}$$

$$u_3(\boldsymbol{x}^*) = 1$$

As is shown by Scarf (1960), this PD dynamic has a unique spatial equilibrium $(1/3, 1/3, 1/3)$, which is however unstable. In fact, it has a limit cycle. This result is robust against any infinitesimal change in the utility functions.

What if there are interactions only within regions as $u_i(\boldsymbol{x}) = u_i(x_i)$? In this case, we can show that there generically exists at least one stable equilibrium.[5] Here, a property is generic if it holds in an open and dense set of the whole topological space of parameters (Mas-Collel, 1985).

Theorem 2 *If $u_i(\boldsymbol{x}) = u_i(x_i)$, then there generically exists at least one asymptotically stable spatial equilibrium for any PD dynamic.*

The proof is contained in Appendix C. This theorem shows if the interactions within regions are much stronger than those among regions ($|u_{ii}| \gg |u_{ij}|$ for all $i \neq j$) as assumed in the previous section, then there always exists at least one asymptotically stable spatial equilibrium, which is either an interior solution or a corner solution. In other words, if the interregional interactions are negligible, we need not to worry about nonexistence of stable equilibrium accompanied with limit cycles or strange attractors.

6 Conclusion

In this paper, we have considered the spatial distribution of economic activities in multiregional dynamics assuming that other variables, such as prices and quantities, are solved as a function of the distribution following the tradition of economic geography a la Krugman (1991). We have focused on positive definite dynamics, which include replicator dynamic and gravity models, where interregional migration is related to interregional utility differentials.

We have obtained stability conditions of the system and shown that strong positive externalities like agglomeration economies instabilize the interior equilibrium, whereas strong negative externalities of congestion stabilize the interior equilibrium (Lemma 1 and Theorem 1).

Imposing the assumption of negligible interregional externalities, we have derived simple stability conditions of spatial interior equilibrium (Proposition 1), which holds for any geographical location of cities, for any speed of adjustments, and for any equilibrium distribution of population. Interpreting the stability conditions, we have illustrated how the multiregional system becomes stable. Similar to the case of interactions among regions, but more neatly, we have shown the effects externalities on the system.

As shown by Scarf (1960), stable spatial equilibria do not necessarily exist in general. However, we have proven that there generically exists a stable equilibrium, for any PD dynamic when each regional utility depends only on its own population (Theorem 2).

[5] Dierker (1974, p.47) intuitively explained that a smooth gradient field on a compact manifold must have at least one locally stable equilibrium.

Appendix A: Proof of Lemma 1

For $i, j = 1, \ldots, n$, let the cofactors of the ij-th entries in matrices F, ∂U and A be $\mathcal{F}_{ij} = (-1)^{i+j}\left|\widetilde{F}_{ij}\right|$, $\mathcal{U}_{ij} = (-1)^{i+j}\left|\widetilde{\partial U}_{ij}\right|$ and $\mathcal{A}_{ij} = (-1)^{i+j}\left|\widetilde{A}_{ij}\right|$ respectively, where $\widetilde{F}_{ij}$, $\widetilde{\partial U}_{ij}$ and $\widetilde{A}_{ij}$ are submatrices formed from F, ∂U and A respectively, by deleting the i-th rows and the j-th columns.

Lemma A. 1 *It holds that $\mathcal{F}_{ij} = \mathcal{F}_{11}$ for all $i, j = 1, \ldots, n$.*

Proof: For $i = j$, we show that

$$\mathcal{F}_{ii} = \begin{vmatrix} f_1 & \cdots & -f_{1,i-1} & -f_{1,i+1} & \cdots & -f_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -f_{1,i-1} & \cdots & f_{i-1} & -f_{i-1,i+1} & \cdots & -f_{i-1,n} \\ -f_{1,i+1} & \cdots & -f_{i-1,i+1} & f_{i+1} & \cdots & -f_{i+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -f_{1n} & \cdots & -f_{i-1,n} & -f_{i+1,n} & \cdots & f_n \end{vmatrix} = \mathcal{F}_{11}$$

holds for $i = 2, \ldots, n$ by the five steps. In $\mathcal{F}_{ii}$, (1) replace the first column with the sum of all the columns; (2) replace the first row with the sum of all the rows; (3) change the signs of all the entries in the first column and the first row; (4) move the first column to the i-th column; and (5) move the first row to the i-th row.

For $i \neq j$, we show the case of $i = 1, j = 2$ only since other cases can be shown in the same way. We have

$$\mathcal{F}_{12} = -\begin{vmatrix} -f_{12} & -f_{23} & \cdots & -f_{2n} \\ -f_{13} & f_3 & \cdots & -f_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ -f_{1n} & -f_{2n} & \cdots & f_n \end{vmatrix} = \begin{vmatrix} f_{12} & -f_{23} & \cdots & -f_{2n} \\ f_{13} & f_3 & \cdots & -f_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{1n} & -f_{2n} & \cdots & f_n \end{vmatrix}$$

$$= \begin{vmatrix} f_1 & f_{13} & \cdots & f_{1n} \\ f_{13} & f_3 & \cdots & -f_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{1n} & -f_{2n} & \cdots & f_n \end{vmatrix} = \begin{vmatrix} f_1 & -f_{13} & \cdots & -f_{1n} \\ -f_{13} & f_3 & \cdots & -f_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ -f_{1n} & -f_{2n} & \cdots & f_n \end{vmatrix} = \mathcal{F}_{22} = \mathcal{F}_{11}.$$

□

Lemma A. 2 *Let the characteristic polynomial $\Psi_A(\lambda) = |A - \lambda E|$. Then,*

$$\Psi_A(0) = 0,$$

$$\Psi'_A(0) = -\sum_{i=1}^{n} \mathcal{A}_{ii} = -\mathcal{F}_{11} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathcal{U}_{ij}.$$

Proof: (i) If $|\partial U| \neq 0$, then the inverse matrix of ∂U is given by

$$\partial U^{-1} = \frac{1}{|\partial U|} \begin{pmatrix} \mathcal{U}_{11} & \mathcal{U}_{21} & \cdots & \mathcal{U}_{n1} \\ \mathcal{U}_{12} & \mathcal{U}_{22} & \cdots & \mathcal{U}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{U}_{1n} & \mathcal{U}_{2n} & \cdots & \mathcal{U}_{nn} \end{pmatrix}.$$

Therefore, we have

$$\begin{aligned} \Psi_A(\lambda) &= |F\partial U - \lambda E| = |F - \lambda \partial U^{-1}||\partial U| \\ &= |\partial U| \begin{vmatrix} f_1 - \mathcal{U}_{11}\lambda/|\partial U| & -f_{12} - \mathcal{U}_{21}\lambda/|\partial U| & \cdots & -f_{1n} - \mathcal{U}_{n1}\lambda/|\partial U| \\ -f_{12} - \mathcal{U}_{12}\lambda/|\partial U| & f_2 - \mathcal{U}_{22}\lambda/|\partial U| & \cdots & -f_{2n} - \mathcal{U}_{n2}\lambda/|\partial U| \\ \vdots & \vdots & \ddots & \vdots \\ -f_{1n} - \mathcal{U}_{1n}\lambda/|\partial U| & -f_{2n} - \mathcal{U}_{2n}\lambda/|\partial U| & \cdots & f_n - \mathcal{U}_{nn}\lambda/|\partial U| \end{vmatrix}, \end{aligned}$$

and hence

$$\begin{aligned} \Psi'_A(0) &= |\partial U| \sum_{i=1}^{n} \begin{vmatrix} f_1 & \cdots & -f_{1,i-1} & -\mathcal{U}_{i1}/|\partial U| & -f_{1,i+1} & \cdots & -f_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -f_{1i} & \cdots & -f_{i-1,i} & -\mathcal{U}_{ii}/|\partial U| & -f_{i,i+1} & \cdots & -f_{in} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -f_{1n} & \cdots & -f_{in} & -\mathcal{U}_{in}/|\partial U| & -f_{i+1,n} & \cdots & f_n \end{vmatrix} \\ &= -\sum_{i=1}^{n}\sum_{j=1}^{n} \mathcal{U}_{ij}\mathcal{F}_{ij} = -\mathcal{F}_{11} \sum_{i=1}^{n}\sum_{j=1}^{n} \mathcal{U}_{ij}, \end{aligned}$$

where the last equality is from Lemma A.1.

(ii) In the case of $|\partial U| = 0$, let $\partial U(\epsilon) = \partial U - \epsilon E$ with $\epsilon > 0$. We have $|\partial U(\epsilon)| = |\partial U - \epsilon E| \neq 0$ if ϵ is not an eigenvalue of ∂U. Since ∂U has a finite number of eigenvalues, $|\partial U(\epsilon)| \neq 0$ holds for sufficiently small ϵ. We know from (i) that the coefficient of λ in the polynomial $|\partial U(\epsilon)F - \lambda E|$ is $-\mathcal{F}_{11} \sum_{i=1}^{n} \sum_{i=1}^{n} \mathcal{U}_{ij}(\epsilon)$, where $\mathcal{U}_{ij}(\epsilon)$ is the cofactor of u_{ij} in $\partial U(\epsilon)$. Since

$$\lim_{\epsilon \to 0} \partial U(\epsilon) = \partial U \qquad \text{and} \qquad \lim_{\epsilon \to 0} \mathcal{U}_{ij}(\epsilon) = \mathcal{U}_{ij},$$

the results of (i) also hold. □

For any non-empty set $T = \{t_1, \ldots, t_k\} \subseteq \{1, 2, \ldots, n\}$, let $\overline{T} = \{1, 2, \ldots, n\} - T$ be its complement set. Define a matrix

$$M_{T,t_i} = \begin{pmatrix} f_{t_1} - \sum_{l \in \overline{T}} f_{t_1 l} & \cdots & -f_{t_1 t_{i-1}} & -f_{t_1 t_{i+1}} & \cdots & -f_{t_1 t_k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -f_{t_1 t_{i-1}} & \cdots & f_{t_{i-1}} - \sum_{l \in \overline{T}} f_{t_{i-1} l} & -f_{t_{i-1} t_{i+1}} & \cdots & -f_{t_{i-1} t_k} \\ -f_{t_1 t_{i+1}} & \cdots & -f_{t_{i-1} t_{i+1}} & f_{t_{i+1}} - \sum_{l \in \overline{T}} f_{t_{i+1} l} & \cdots & -f_{t_{i+1} t_k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -f_{t_1 t_k} & \cdots & -f_{t_{i-1} t_k} & -f_{t_{i+1} t_k} & \cdots & f_{t_k} - \sum_{l \in \overline{T}} f_{t_k l} \end{pmatrix}.$$

for any $t_i \in T$ if T contains at least two elements. If T contains only one element, then we set $\det M_T = 1$ for convenience.

We show that $\det M_{T,t_i} = \det M_{T,t_1}$ holds for all $i = 2, \dots, n$ by the following five steps: (1) replace the first column with the sum of all the columns; (2) replace the first row with the sum of all the rows; (3) change the signs of all the entries in the first column and the first row; (4) move the first column to the i-th column; and (5) move the first row to the i-th row. Therefore, $\det M_{T,t_i}$ does not depend on t_i, and hence we sometimes write $\det M_T$.

Lemma A. 3

$$\begin{vmatrix} 0 & -y_2 & -y_3 & \cdots & -y_n \\ -y_2 & f_2 & -f_{23} & \cdots & -f_{2n} \\ -y_3 & -f_{23} & f_3 & \cdots & -f_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -y_n & -f_{2n} & -f_{3n} & \cdots & f_n \end{vmatrix} = - \sum_{\emptyset \neq T \subseteq \{2,\cdots,n\}} \left(\sum_{t \in T} y_t \right)^2 \det M_T \det M_{\overline{T}} < 0 \qquad (13)$$

holds for any real numbers $y_2, \dots, y_n$*, where* $\overline{T} = \{1, \dots, n\} - T$.

Proof: Since $1 \in \overline{T}$ holds for any $T \subseteq \{2, \dots, n\}$, we know that $\overline{T} \neq \emptyset$, and hence $\det M_T$ and $\det M_{\overline{T}}$ are well-defined. We first show the inequality by proving that for any set $T \subset \{1, \dots, n\}$ with $T \neq \emptyset$ and $\overline{T} \neq \emptyset$, it holds that $\det M_T > 0$. In fact, if T contains only one element, then $\det M_T = 1 > 0$. If T contains at least two elements, then M_T is a symmetric matrix with positive dominant diagonals because $\overline{T} \neq \emptyset$. By Theorem 4.C.2 in Takayama (1985), all the eigenvalues of M_T have positive real parts. Since M_T is positive definite, $\det M_T > 0$ holds.

We next show that the equality holds by examining all the coefficients of $y_i y_j$ $(i, j = 2, \dots, n)$ of both sides one by one. For illustration, consider the coefficients of y_2^2 and $y_2 y_3$. The coefficients of $-y_2^2$ in the LHS of (13) is

$$\begin{vmatrix} f_3 & -f_{34} & \cdots & -f_{3n} \\ -f_{34} & f_4 & \cdots & -f_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ -f_{3n} & -f_{4n} & \cdots & f_n \end{vmatrix}. \tag{14}$$

For any set $T = \{2, t_1, \ldots, t_k\} \subseteq \{2, \ldots, n\}$, the terms of (14) not containing all $f_{t,l}$ $(t \in T, l \in \overline{T})$ are

$$\begin{vmatrix} M_{T,2} & \mathbf{0} \\ \mathbf{0} & M_{\overline{T},1} \end{vmatrix} = \det M_T \det M_{\overline{T}},$$

where $\mathbf{0}$ is a zero matrix with a suitable size. Therefore, (14) coincides with the coefficient of $-y_2^2$ in the RHS of (13). The coefficient of $-2y_2y_3$ in the LHS is

$$\begin{vmatrix} f_{23} & -f_{34} & \cdots & -f_{3n} \\ f_{24} & f_4 & \cdots & -f_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{2n} & -f_{4n} & \cdots & f_n \end{vmatrix}. \tag{15}$$

For any set $T = \{2, 3, t_1, \ldots, t_k\} \subseteq \{2, 3, \ldots, n\}$, we have

$$\begin{vmatrix} f_{23} & -f_{3t_1} & \cdots & -f_{3t_k} \\ f_{2t_1} & f_{t_1} - \sum_{l \in \overline{T}} f_{t_1} l & \cdots & -f_{t_1 t_k} \\ \vdots & \vdots & \ddots & \vdots \\ f_{2t_k} & -f_{t_1 t_k} & \cdots & f_{t_k} - \sum_{l \in \overline{T}} f_{t_k} l \end{vmatrix} = - \begin{vmatrix} -f_{23} & -f_{3t_1} & \cdots & -f_{3t_k} \\ -f_{2t_1} & f_{t_1} - \sum_{l \in \overline{T}} f_{t_1} l & \cdots & -f_{t_1 t_k} \\ \vdots & \vdots & \ddots & \vdots \\ -f_{2t_k} & -f_{t_1 t_k} & \cdots & f_{t_k} - \sum_{l \in \overline{T}} f_{t_k} l \end{vmatrix}$$

$$= - \begin{vmatrix} -f_3 + \sum_{l \in \overline{T}} f_{3l} & -f_{3t_1} & \cdots & -f_{3t_k} \\ f_{3t_1} & f_{t_1} - \sum_{l \in \overline{T}} f_{t_1} l & \cdots & -f_{t_1 t_k} \\ \vdots & \vdots & \ddots & \vdots \\ f_{3t_k} & -f_{t_1 t_k} & \cdots & f_{t_k} - \sum_{l \in \overline{T}} f_{t_k l} \end{vmatrix} = \det M_{T,2}.$$

Therefore, the terms of (15) without containing all $f_{t,l}$ $(t \in T, l \in \overline{T})$ are

$$\begin{vmatrix} M_{T,2} & \mathbf{0} \\ \mathbf{0} & M_{\overline{T},1} \end{vmatrix} = \det M_T \det M_{\overline{T}}.$$

Hence, (15) coincides with the coefficient of $-2y_2y_3$ in the RHS of (13). □

Lemma A. 4 *(i) If*

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\mathcal{U}_{ij} \neq 0, \tag{16}$$

then 0 *is a simple eigenvalue of matrix* A.

If $|\partial U| \neq 0$ *and (16) is violated, then* 0 *is a 2-multiple eigenvalue.*

Proof: From A.2, we know that $\lambda = 0$ is a simple eigenvalue of A if and only if (16) holds. Furthermore, if $|\partial U| \neq 0$ and (16) does not hold, then

$$\begin{aligned}
\Psi_A(\lambda) &= |F\partial U - \lambda E| = |F - \lambda \partial U^{-1}||\partial U| \\
&= |\partial U| \begin{vmatrix} f_1 - \mathcal{U}_{11}\lambda/|\partial U| & -f_{12} - \mathcal{U}_{21}\lambda/|\partial U| & \cdots & -f_{1n} - \mathcal{U}_{n1}\lambda/|\partial U| \\ -f_{12} - \mathcal{U}_{12}\lambda/|\partial U| & f_2 - \mathcal{U}_{22}\lambda/|\partial U| & \cdots & -f_{2n} - \mathcal{U}_{n2}\lambda/|\partial U| \\ \vdots & \vdots & \ddots & \vdots \\ -f_{1n} - \mathcal{U}_{1n}\lambda/|\partial U| & -f_{2n} - \mathcal{U}_{2n}\lambda/|\partial U| & \cdots & f_n - \mathcal{U}_{nn}\lambda/|\partial U| \end{vmatrix} \\
&= |\partial U| \begin{vmatrix} -\sum_{i=1}^{n}\mathcal{U}_{i1}\lambda/|\partial U| & -f_{12} - \mathcal{U}_{21}\lambda/|\partial U| & \cdots & -f_{1n} - \mathcal{U}_{n1}\lambda/|\partial U| \\ -\sum_{i=1}^{n}\mathcal{U}_{i2}\lambda/|\partial U| & f_2 - \mathcal{U}_{22}\lambda/|\partial U| & \cdots & -f_{2n} - \mathcal{U}_{n2}\lambda/|\partial U| \\ \vdots & \vdots & \ddots & \vdots \\ -\sum_{i=1}^{n}\mathcal{U}_{in}\lambda/|\partial U| & -f_{2n} - \mathcal{U}_{2n}\lambda/|\partial U| & \cdots & f_n - \mathcal{U}_{nn}\lambda/|\partial U| \end{vmatrix} \\
&= |\partial U| \begin{vmatrix} 0 & -\sum_{j=1}^{n}\mathcal{U}_{2j}\lambda/|\partial U| & \cdots & -\sum_{j=1}^{n}\mathcal{U}_{nj}\lambda/|\partial U| \\ -\sum_{i=1}^{n}\mathcal{U}_{i2}\lambda/|\partial U| & f_2 - \mathcal{U}_{22}\lambda/|\partial U| & \cdots & -f_{2n} - \mathcal{U}_{n2}\lambda/|\partial U| \\ \vdots & \vdots & \ddots & \vdots \\ -\sum_{i=1}^{n}\mathcal{U}_{in}\lambda/|\partial U| & -f_{2n} - \mathcal{U}_{2n}\lambda/|\partial U| & \cdots & f_n - \mathcal{U}_{nn}\lambda/|\partial U| \end{vmatrix} \\
&= |\partial U| \begin{vmatrix} 0 & -\sum_{j=1}^{n}\mathcal{U}_{2j}/|\partial U| & \cdots & -\sum_{j=1}^{n}\mathcal{U}_{nj}/|\partial U| \\ -\sum_{i=1}^{n}\mathcal{U}_{i2}/|\partial U| & f_2 - \mathcal{U}_{22}\lambda/|\partial U| & \cdots & -f_{2n} - \mathcal{U}_{n2}\lambda/|\partial U| \\ \vdots & \vdots & \ddots & \vdots \\ -\sum_{i=1}^{n}\mathcal{U}_{in}/|\partial U| & -f_{2n} - \mathcal{U}_{2n}\lambda/|\partial U| & \cdots & f_n - \mathcal{U}_{nn}\lambda/|\partial U| \end{vmatrix} \lambda^2.
\end{aligned}$$

Then, the coefficient of λ^2 in the characteristic polynomial is given by

$$|\partial U| \begin{vmatrix} 0 & -\sum_{j=1}^{n}\mathcal{U}_{2j}/|\partial U| & \cdots & -\sum_{j=1}^{n}\mathcal{U}_{nj}/|\partial U| \\ -\sum_{i=1}^{n}\mathcal{U}_{i2}/|\partial U| & f_2 & \cdots & -f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\sum_{i=1}^{n}\mathcal{U}_{in}/|\partial U| & -f_{2n} & \cdots & f_n \end{vmatrix}, \tag{17}$$

which is nonzero by (13). Therefore, the multiplicity of eigenvalue $\lambda = 0$ is 2.

Proposition A. 1 *If* $a_{ij} > 0$ *holds for all* $i, j = 1, \ldots, n, i \neq j$*, then one eigenvalue of* A *is* 0 *and all other eigenvalues have negative real parts.*

Proof: Let the Geršgorin disks of matrix A be

$$D^i = \left\{ \text{complex numbers } z \;\middle|\; |z - a_{ii}| \leq \sum_{j=1, j\neq i}^{n} |a_{ji}| \right\}, \quad i = 1, \dots, n.$$

We have the following equalities

$$\sum_{j\neq i} |a_{ji}| = \left| \sum_{j\neq i} a_{ji} \right| = |a_{ii}| = -a_{ii},$$

where the first equality is due to $a_{ij} > 0$ for all $j \neq i$, the second and the third are from $\sum_{i=1}^{n} a_{ij} = 0$. Therefore, all the eigenvalues of A have nonpositive real parts and any pure imaginary number cannot become an eigenvalue from Geršgorin theorem (Theorem 6.1.1 in Horn and Johnson (1985)).

Consider matrix $\tilde{A}_{nn}^T$, which is the transpose of $\tilde{A}_{nn}$. From $\sum_{i=1}^{n} a_{ij} = 0$, we have

$$\tilde{A}_{nn}^T \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = - \begin{pmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{n,n-1} \end{pmatrix}. \tag{18}$$

Since $a_{ii} < 0$ and $a_{ij} > 0$ hold for all $i, j = 1, \dots, n, i \neq j$, (18) implies that there exists an $\lambda \geq 0$ such that $\tilde{A}_{nn}^T \lambda < 0$. Hence, $\tilde{A}_{nn}^T$ is Hicksian by Theorem 4.D.3 in Takayama (1985), so $(-1)^{n-1}\mathcal{A}_{nn} = (-1)^{n-1}|\tilde{A}_{nn}| = (-1)^{n-1}|\tilde{A}_{nn}^T| > 0$. Similarly, we have $(-1)^{n-1}\mathcal{A}_{ii} > 0$ for all $i = 1, \dots, n-1$. Thus, $(-1)^{n-1}\sum_{i=1}^{n}\mathcal{A}_{ii} > 0$, which is the coefficient of λ in the characteristic polynomial $\Psi_A(\lambda)$ from Lemma A.2. This implies that 0 is a simple eigenvalue of A. □

Proposition A. 2 *If*
(i) (5) *holds;*
(ii) matrix A has $l(\geq 2)$ positive and $n - l$ non-positive diagonal entries; and
(iii) for each i, a_{ji} do not have different signs for all $j = 1, \dots, n, j \neq i$;
then there is an eigenvalue of A with a positive real part.

Proof: For $k = 1, 2, \dots$, let

$$A(k) = \begin{pmatrix} (1+1/k)a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & (1+1/k)a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & (1+1/k)a_{nn} \end{pmatrix}.$$

Then $\lim_{k\to\infty} A(k) = A$. The following n sets $D^i(k)$ are the Geršgorin disks of matrix $A(k)$.

$$D^i(k) = \left\{ \text{complex numbers } z \;\middle|\; |z - (1+1/k)a_{ii}| \leq \sum_{j=1, j\neq i}^{n} |a_{ji}| = |a_{ii}| \right\}, \quad i = 1, \dots, n, \tag{19}$$

where the last equality is from condition (iii).

Condition (ii) means that matrix A has $l(\geq 2)$ positive and $n-l$ nonpositive diagonal entries. Without loss of generality, suppose that $a_{11} > 0, \ldots, a_{ll} > 0$ and $a_{l+1,l+1} \leq 0, \ldots, a_{nn} \leq 0$. Then, for any finite k, the disks $D^1(k), \ldots, D^l(k)$ are separated from other disks $D^{l+1}(k), \ldots, D^n(k)$. By Geršgorin theorem, there exist l eigenvalues $z^1(k), \ldots, z^l(k)$ of $A(k)$ in $\cup_{j=1}^l D^j(k)$. Since $\{z^1(k)\}, \ldots, \{z^l(k)\}$ are bounded, they have convergent subsequences. For convenience, we suppose that they themselves are convergent and let

$$\lim_{k\to\infty} z^1(k) = z_0^1, \ldots, \lim_{k\to\infty} z^l(k) = z_0^l.$$

Then, $z_0^1, \ldots, z_0^l$ are either 0 or some complex numbers with positive real parts. Furthermore, we know from

$$|A - z_0^1 E| = \lim_{k\to\infty} |A(k) - z^1(k)E| = 0, \ldots, |A - z_0^l E| = \lim_{k\to\infty} |A(k) - z_k^l E| = 0,$$

that $z_0^1, \ldots, z_0^l$ are eigenvalues of A. Since $\lambda = 0$ is a simple root of $\Psi_A(\lambda) = 0$ from Lemma A.4, $z_0^1, \ldots, z_0^l$ cannot become zero simultaneously, implying that there exists at least one eigenvalue with a positive real part. □

Proposition A. 3 *If*
(i) matrix A has one positive and $n-1$ nonpositive diagonal entries; and
(ii) for given i, a_{ij} do not have different signs for all $j = 1, \ldots, n, j \neq i$;
then one eigenvalue of A is 0 and all other eigenvalues have negative real parts when

$$(-1)^n \sum_{i=1}^n \sum_{j=1}^n \mathcal{U}_{ij} < 0, \tag{20}$$

and one eigenvalue is positive when the inequality in (20) is reversed.

Proof: Without loss of generality, let $a_{11} > 0$ and $a_{ii} \leq 0$ for all $i = 2, \ldots, n$. Then, the Geršgorin disk $D^1(k)$ of (19) is separated from the other Geršgorin disks, and so $A(k)$ has one eigenvalue $z^1(k)$ with a positive real part, while all the other eigenvalues have negative real parts.

If (20) holds, then $(-1)^{n-1}\Psi'_A(0) > 0$ from Lemma A.2. Since $(-1)^{n-1}\Psi'_A(0)$ equals the product of all non-zero eigenvalues, the number of non-zero eigenvalues with positive real parts should be even, and hence there is indeed no eigenvalue having a positive real part. Since any Geršgorin disk does not cover pure imaginary numbers (i.e., zero real parts), all non-zero eigenvalues have negative real parts.

If the reverse inequality in (20) holds, then $(-1)^{n-1}\Psi'_A(0) < 0$. Since it evidently holds that $\lim_{\lambda\to+\infty}(-1)^{n-1}\Psi_A(\lambda) = +\infty$, equation $\Psi_A(\lambda) = 0$ has at least one positive root, which is an eigenvalue of A. By Geršgorin theorem, A has only one positive eigenvalue. □

Appendix B: Proof of Proposition 1

First, we consider the case of $u_{ii} \neq 0$ for all i. Then, we have

$$a_{ij} = \begin{cases} \sum_{k=1,k\neq i}^{n} u_{ii} f_{ik} & \text{for } i = j, \\ -u_{jj} f_{ij} & \text{for } i \neq j, \end{cases}$$

and hence, $a_{ij} > 0$ iff $u_{jj} < 0$ for all $i \neq j$. Furthermore, since

$$\mathcal{U}_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ \frac{1}{u_{ii}} \prod_{k=1}^{n} u_{kk} & \text{if } i = j, \end{cases}$$

we get

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \mathcal{U}_{ij} = \prod_{k=1}^{n} (-u_{kk}) \sum_{i=1}^{n} \frac{1}{u_{ii}}.$$

Thus, Proposition 1 directly follows from Lemma 1 for $u_{11} \neq 0$ and $u_{22} \neq 0$.

Second, in the case of $u_{11} = 0$ and $u_{22} < 0$, matrix A takes the following form:

$$\begin{pmatrix} 0 & + & + & \cdots & + \\ 0 & - & + & \cdots & + \\ 0 & + & - & \cdots & + \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & + & + & \cdots & - \end{pmatrix}.$$

Obviously, $(-1)^{n-1}\mathcal{A}_{ii} = 0$ holds for $i = 2, \ldots, n$, while $(-1)^{n-1}\mathcal{A}_{11} > 0$ holds by the similar proof of Proposition A.1. Hence, Proposition 1 (i) is shown in the case of $u_{11} = 0$.

Finally, Proposition 1 (iv) with $a_{11} > a_{22} = 0$ is shown by checking the multiplicity of 0 eigenvalue. □

Appendix C: Proof of Theorem 2

According to the topology of the $\mathcal{C}^1$ uniform convergence, functions f_k converge to function f if and only if $f_k - f$ and $f'_k - f'$ converge uniformly to zero.

Given $u_1(x_1), \ldots, u_n(x_n)$, let $\mathcal{P}_u(\boldsymbol{x}) = -\sum_{i=1}^{n} \int_0^{x_i} u_i(t)dt$. The Hessian matrix of this function at $\boldsymbol{x}^* = (x_1^*, \ldots, x_n^*)$ is $\text{diag}[-u'_1(x_1^*), \ldots, -u'_n(x_n^*)]$. For convenience, we introduce the following $(m+1) \times (m+1)$ bordered matrix

$$H_u(\boldsymbol{x}_m) = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & -u'_1(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & -u'_m(x_m) \end{pmatrix},$$

where $m \le n$. The bordered principal minor is

$$\begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & -u_1'(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & -u_k'(x_k) \end{vmatrix} = (-1)^k \sum_{i=1}^{k} \prod_{j=1, j\neq i}^{k} u_j'(x_j), \quad k = 2, \ldots, n. \tag{21}$$

The necessary (resp. sufficient) conditions for an interior point $\boldsymbol{x}^* = (x_1^*, \ldots, x_n^*)$ to be a local minimizer of $\mathcal{P}_u(\boldsymbol{x})$ subject to $\sum_{i=1}^n x_i = 1$ and $x_i \ge 0$ are that $u_i(x_i^*) = u_j(x_j^*)$ for all $i, j = 1, \ldots, n$ and the bordered principal minors (21) at $\boldsymbol{x}^*$ are nonpositive (resp. negative) for all $k = 1, \ldots, n$ (Theorem 1.E.17 in Takayama (1985)). The sufficient condition is equivalent to the stability condition of Proposition 1.

More generally, sufficient conditions for $\boldsymbol{x}^* = (x_1^*, \ldots, x_{n_1}^*, 0, \ldots, 0) \equiv (\boldsymbol{x}_{n_1}^*, \mathbf{0})$ with $1 \le n_1 \le n$ to be a local minimizer of $\mathcal{P}_u(\boldsymbol{x})$ are

(i) the utility of a non-empty region is higher than that of an empty region:

$$u_1(x_1^*) = \cdots = u_{n_1}(x_{n_1}^*) > u_{n_1+j}(0) \quad \text{for } j = 1, \ldots, n - n_1, \tag{22}$$

(ii) the bordered principal minors (21) of bordered matrix $H_u(\boldsymbol{x}_{n_1}^*)$ are negative for all $k = 1, \ldots, n_1$.

These two conditions also ensure that $\boldsymbol{x}^*$ is a stable corner equilibrium by the arguments in Zeng (2002). Therefore, the following lemma establishes the existence of a stable equilibrium.

Lemma C. 1 *Given functions $u_i(x_i)$, there exist $n_1(\le n)$ and a local minimizer $\boldsymbol{x}^* = (\boldsymbol{x}_{n_1}^*, \mathbf{0})$ of $\mathcal{P}_u(\boldsymbol{x})$with $x_i^* > 0$ for $i = 1, \ldots, n_1$ subject to $\sum_{i=1}^n x_i = 1$ (after necessary exchanges in the order of components). Furthermore, given the topology of $\mathcal{C}^1$ uniform convergence, the above properties (i) and (ii) hold generically.*

Proof: Since $\mathcal{P}_u(\boldsymbol{x})$ is continuous over the compact set

$$X = \{(x_1, \ldots, x_n)' | \ x_i \ge 0, \sum_{i=1}^n x_i = 1\},$$

it should have a local minimizer $\boldsymbol{x}^* = (\boldsymbol{x}_{n_1}^*, \mathbf{0})$. What we need to show is the genericity of properties (i) and (ii).

First, we show that the set of functions u_i satisfying (i) and (ii) is dense in the whole functional space. In fact, for any function $u_i(t)$ with a local minimizer $\boldsymbol{x}^* = (x_1^*, \ldots, x_{n_1}^*, 0, \ldots, 0)'$ of $\mathcal{P}_u(\boldsymbol{x})$ satisfying (i) and (ii), we can construct the following sequence of functions

$$u_i^k(x_i) = \begin{cases} u_i(x_i) - \frac{x_i - x_i^*}{k}, & \text{for } i = 1, \ldots, n_1, \\ u_i(x_i) - \frac{1}{k}, & \text{for } i = n_1 + 1, \ldots, n. \end{cases}$$

Since $\boldsymbol{x}^*$ is a local minimizer of $\mathcal{P}_u(\boldsymbol{x})$, we have

$$u_1(x_1^*) = \cdots = u_{n_1}(x_{n_1}^*) \geq u_{n_1+j}(x_{n_1+j}^*) \qquad \text{for } j = 1, \ldots, n - n_1.$$

Therefore, $u^k(\boldsymbol{x})$ satisfies (i) for each k. Again, since $\boldsymbol{x}^*$ is a local minimizer of $\mathcal{P}_u(\boldsymbol{x})$, all the principal minors of (21) should be nonpositive. However, we have $(u_i^k)'(x_i) = u_i'(x_i) - 1/k$. By calculating the principal minors of bordered matrix $H_{u^k}(\boldsymbol{x}_{n_1})$, we know that u^k and $\mathcal{P}_{u^k}(\boldsymbol{x})$ also satisfy (ii).

Second, we show that the set of functions u_i satisfying (i) and (ii) is open in the whole functional space. Suppose that functions $u_i(t)$ satisfy (i) and (ii). Let $\mathcal{R}_0^{n_1} = \{\boldsymbol{x} = (x_1, \ldots, x_{n_1})' \in \mathcal{R}^{n_1} | \sum_{i=1}^{n_1} x_i = 0\}$. Because functions $u_i(x_i)$ and $u_i'(x_i)$ are continuous, there exists a ball $B(r)$ of $\mathcal{R}_0^{n_1}$ with center 0 and radius r such that all the principal minors of bordered matrix $H_v(\boldsymbol{x}_{n_1})$ are negative for all $\boldsymbol{x}_{n_1} \in B(\boldsymbol{x}_{n_1}^*, r) = \boldsymbol{x}_{n_1}^* + B(r)$ and functions $v_1(x_1), \ldots, v_n(x_n)$ such that $||v_i - u_i|| \equiv \max_{t\in[0,1]}(|v_i(t) - u_i(t)|) < r$ and $||v_i' - v_i'|| < r$. Let r be sufficiently small such that

$$\mathcal{P}_u(\boldsymbol{x}_{n_1}^*) < \mathcal{P}_u(\boldsymbol{y}_{n_1}), \qquad \forall \boldsymbol{y}_{n_1} \in B(\boldsymbol{x}_{n_1}^*, r) - \{\boldsymbol{x}_{n_1}^*\}.$$

Denote the surface of ball $B(\boldsymbol{x}_{n_1}^*, r)$ by $S(\boldsymbol{x}_{n_1}^*, r)$. Then, we have

$$\epsilon_2 = \min_{\boldsymbol{y}_{n_1} \in S(\boldsymbol{x}_{n_1}^*, r)} \mathcal{P}_u(\mathbf{y}_{n_1}) - \mathcal{P}_u(\boldsymbol{x}_{n_1}^*) > 0.$$

Finally, from property (i),

$$\epsilon_3 = u_1(x_1^*) - \max_{j=1,\ldots,n-n_1} u_{n_1+j}(0) > 0. \tag{23}$$

holds. Let

$$\epsilon = \min\{r, \epsilon_1, \epsilon_2, \epsilon_3\}/2(> 0). \tag{24}$$

and let $v_i(t)$ be any function in the neighborhood of $u_i(t)$ satisfying $||v_i - u_i|| < \epsilon$. Then,

$$\begin{aligned}
\mathcal{P}_v(\boldsymbol{x}_{n_1}^*) &= -\sum_{i=1}^{n_1} \int_0^{x_i^*} v_i(t)dt \\
&< -\sum_{i=1}^{n_1} \int_0^{x_i^*} u_i(t)dt + \epsilon \sum_{i=1}^{n_1} x_i^* \\
&= -\epsilon_2 + \min_{\boldsymbol{y}_{n_1} \in S(\boldsymbol{x}_{n_1}^*, r)} \mathcal{P}_u(\boldsymbol{y}_{n_1}) + \epsilon \\
&= -\min_{\boldsymbol{y}_{n_1} \in S(\boldsymbol{z}_{n_1}^*, r)} \sum_{i=1}^{n_1} \int_0^{y_i} u_i(t)dt - \epsilon_2 + \epsilon \\
&< -\min_{\boldsymbol{y}_{n_1} \in S(\boldsymbol{z}_{n_1}^*, r)} \sum_{i=1}^{n_1} \int_0^{y_i} v_i(t)dt - \epsilon_2 + 2\epsilon \\
&\leq \min_{\boldsymbol{y}_{n_1} \in S(\boldsymbol{z}_{n_1}^*, r)} \mathcal{P}_v(\boldsymbol{x}_{n_1}).
\end{aligned}$$

Thus, $\mathcal{P}_v(\boldsymbol{x}_{n_1})$ should have an interior minimum $\boldsymbol{x}^\sharp_{n_1}$ in $B(\boldsymbol{x}^*_{n_1}, r)$. We know from (23) and (24) that $v_1, \ldots, v_n$ and $\boldsymbol{x}^\sharp_{n_1}$ satisfies (i). In addition, since all the principal minors of bordered matrix $H_v(\boldsymbol{x}_{n_1})$ are negative in $B(\boldsymbol{x}^*_{n_1}, r)$, (ii) also holds. □

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