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Abstract

This paper explains how an exogenous borrowing constraint affects the fertility rate in an overlapping-generations model for four-period-lived agents. Young agents face a borrowing constraint when they decide how many children they want to have. Relaxing the constraint increases borrowing by a young agent, who will thus have more children. At the same time, if a young agent borrows more, the interest rate will increase and labor wage will decrease, because of a decrease in physical capital stock and an increase in labor. This makes a young agent's debt much heavier, and a young agent will have fewer children. Considering these two effects together, relaxing the borrowing constraint does not necessarily have a positive effect on the fertility rate. In addition, the paper discusses the effect of the borrowing constraint on welfare.

Keywords: Fertility rate, exogenous borrowing constraint, overlapping-generations model

JEL Classification: J13, D91

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1 Introduction

Raising a child is costly. For instance, it costs \$245,000 in the U.S.¹ and £230,000 in the U.K.² to raise a child till ages 18 and 21, respectively. Such a large amount of money could explain why young people in the developed countries, whose earnings are usually not high, do not want to (or cannot) have many children. Given that people's earnings grow over the years, a financial market that helps people transfer money inter-temporarily can incentivize young people to have more children. This paper investigates how such a financial market affects the fertility rate by focusing on the role of an exogenously given borrowing constraint in an overlapping-generations model.

This paper considers an overlapping-generations (OLG) model for four-period-lived agents. It extends the model in Jappelli and Pagano (1994) by endogenizing the population growth rate.³ In the first period, the agent is a child and makes no decisions. In the second period, the *young* agent determines how many children he or she will have. The young agent does not have enough wealth to raise children, which is a costly affair. Thus, the agent borrows in the capital market, and will repay the debt in the next period. The agent faces a borrowing constraint in the capital market. The amount that a young agent can borrow is bounded above. This upper bound is determined by some ratio of the next-period labor income, denoted by $\mu \in (0, 1]$. In the third period, the *middle-aged* agent provides labor inelastically and earns labor income. With this labor income, the agent determines the amounts of consumption and savings, and repays the debt. In the fourth period, the *old* agent retires, receives interest income, and spends all of it for consumption. An agent derives utility from the number of children and consumption in middle and old age. A representative firm produces output using labor and capital stock. Under this setting, I characterize a standard perfect-foresight competitive equilibrium and analyze how the borrowing constraint, μ , affects variables such as fertility rate and welfare in the long run.

The main finding is that the fertility rate in the long run is hump-shaped in μ when μ is not too high

¹See <http://money.cnn.com/2014/08/18/pf/child-cost/>.

²See <http://www.telegraph.co.uk/news/uknews/11360819/Average-cost-of-raising-a-child-in-UK-230000.html>.

³In Jappelli and Pagano (1994), an agent lives for three periods, and the population growth rate is exogenously given.

and constant when μ is sufficiently large. To understand this result, let us consider two effects caused by a relaxation of the borrowing constraint (an increase in μ). The first effect is a positive effect on the fertility rate. An increase in μ , which implies that a young agent can borrow more, raises the fertility rate. The second effect is a negative effect on the fertility rate. If a young agent borrows more, the physical capital stock will decrease because less savings are used for capital accumulation. At the same time, more children imply more labor in the economy. This raises the interest rate and lowers labor income. These imply that borrowing when young will turn into a huge burden in middle age. Therefore, a young agent does not have many children. Whether the fertility rate increases or not in μ depends on these two effects. If the first effect dominates the second effect, then the fertility rate increases as the borrowing constraint is relaxed; if the second effect dominates the first effect, the fertility rate drops as μ increases. Once μ reaches a sufficiently high ratio, the borrowing constraint is not binding anymore. Thus, the fertility rate does not change even if μ increases.

Extensive research has analyzed the fertility rate in a general equilibrium framework (e.g., Eckstein and Wolpin (1985); Becker and Barro (1988); Barro and Becker (1989)). Although many studies have investigated the reasons behind the drop in fertility rate observed in many developed countries, to the best of my knowledge, only one paper explains the relationship between the fertility rate and the imperfect capital market.⁴ Likewise, Filoso and Papagni (2015), were motivated by the same objective as was this study. They show that the relaxed borrowing constraint will increase the fertility rate in a partial equilibrium.⁵ Moreover, they empirically show that improved access to credit increases fertility with an elasticity of around 30%. As they claimed, their analysis is not a general equilibrium analysis. Thus, one important contribution of this paper is that it investigates the effects of the borrowing constraint on the fertility in a dynamic general equilibrium setting. Furthermore, real data show an ambiguous relationship between the borrowing constraint and the fertility rate, which can be explained by the general equilibrium effect found in this paper. Papagni (2006) also examines the relationship between the imperfect capital

⁴Zhang and Zhang (1998) and Wigger (1999) investigate the possibility that the expansion of pay-as-you-go social security lowers the fertility rate.

⁵They also consider, in addition to the borrowing constraint, the effects of the saving constraint that restricts savings by an agent.

market and fertility, although the imperfect capital market affects the amount of human capital investment by parents.

The remainder of this paper is organized as follows. Section 2 describes the economic model, and Section 3 characterizes the equilibrium. Section 4 studies the comparative statics, and Section 5 concludes the study.

2 Model

Time is discrete and continues forever, $t = 1, 2, \dots$. An agent lives for four periods: childhood, young age, middle age, and old. The agent does not make any decisions in childhood. Let N_t be the population of *young* agents in period t . An agent born in period $t - 1$ is a young agent in period t . A young agent decides how many children he or she will have, and eventually raises them. The cost of rearing one child is denoted by $\phi > 0$. Thus, if a young agent in period t has n_t children, then the cost of rearing them is ϕn_t . In addition, the population of agents born in period t is $N_{t+1} = n_t N_t$, which is also the population of young agents in period $t + 1$. Suppose that a young agent does not have enough wealth and has to borrow money to have and raise children. For analytical tractability, I assume that the young agent has no wealth. Furthermore, considering that the capital market is imperfect, the amount a young agent in period t can borrow is restricted by the following borrowing constraint:

$$r_{t+1} \phi n_t \leq \mu w_{t+1}, \quad (1)$$

where r_{t+1} and w_{t+1} are the interest rate and the real wage in period $t + 1$, respectively, and $\mu \in (0, 1]$.

A middle-aged agent in period $t + 1$ supplies labor inelastically, receives labor income, consumes, saves for the next period, and repays his or her debt. Thus, the budget constraint for a middle-aged agent in period $t + 1$ is

$$c_{t+1} + s_{t+1} + r_{t+1} \phi n_t \leq w_{t+1}, \quad (2)$$

where c_{t+1} and s_{t+1} are a middle-aged agent's consumption and savings, respectively. When an agent becomes old in period $t + 2$, he or she receives interest income, and spends all of it for consumption. Thus, the budget constraint for an old agent in period $t + 2$ is

$$d_{t+2} \leq r_{t+2}s_{t+1}, \quad (3)$$

where d_{t+2} is the amount of consumption for an old agent. From Equations (2) and (3), the lifetime budget constraint for an agent born in period $t - 1$ is

$$c_{t+1} + \frac{d_{t+2}}{r_{t+2}} + r_{t+1}\phi n_t \leq w_{t+1}. \quad (4)$$

An agent born in period $t - 1$ derives utility from the number of children, as well as consumption in period $t + 1$ and period $t + 2$. The lifetime utility function is

$$U(n_t, c_{t+1}, d_{t+2}) := \ln(n_t) + \beta \ln(c_{t+1}) + \gamma \ln(d_{t+2}), \quad (5)$$

where $\beta > 0$ and $\gamma > 0$ are weights on utilities from consumption relative to the utility from having children. Thus, an agent born in period $t - 1$ chooses n_t , c_{t+1} , s_{t+1} , and d_{t+2} so as to maximize his or her lifetime utility subject to Equations (1) and (4).

The representative firm's production function is expressed as

$$Y_{t+1} = zF(K_{t+1}, L_{t+1}) := zK_{t+1}^\alpha L_{t+1}^{1-\alpha},$$

where $\alpha \in (0, 1)$, K_{t+1} and L_{t+1} are the aggregate capital stock and aggregate labor, respectively, and $z > 0$ is TFP. Given the real wage, w_{t+1} , and the real rental rate of capital, r_{t+1} , the firm's profit in period $t + 1$ is

$$zK_{t+1}^\alpha L_{t+1}^{1-\alpha} - r_{t+1}K_{t+1} - w_{t+1}L_{t+1}.$$

I assume that capital is fully depreciated after production. Let $k_{t+1} := K_{t+1}/L_{t+1}$ denote capital per unit of labor in period $t + 1$.

The paper analyzes a *perfect foresight competitive equilibrium*, defined as follows:

Definition 2.1. Given $N_0 > 0$, $n_0 > 0$, and $s_0 > 0$, an **equilibrium** consists of a consumption sequence, $(d_t, (c_t, d_{t+1})_{t=1}^\infty)$, a sequence of fertility, $(n_t)_{t=1}^\infty$, a sequence of savings, $(s_t)_{t=1}^\infty$, a sequence of inputs for production, $(K_t, L_t)_{t=1}^\infty$, and a sequence of prices $(w_t, r_t)_{t=1}^\infty$ such that

1. For all $t \geq 1$, given prices, a young agent chooses (n_t, c_t, d_{t+1}) to maximize Equation (5) subject to Equations (1) and (4).
2. Given prices (w_1, r_2) and $n_0 > 0$, the initial middle-aged agent solves

$$\begin{aligned} \max_{c_1, n_1, d_2} \quad & \beta \ln(c_1) + \gamma \ln(d_2) \\ \text{s.t.} \quad & c_1 + \frac{d_2}{r_2} + r_1 \phi n_0 \leq w_1, \end{aligned}$$

where $r_1 \phi n_0 \leq \mu w_1$ is satisfied.

3. For the initial old, $d_1 = r_1 s_0$.
4. In each period t , prices are determined by $w_t = (1 - \alpha) z k_t^\alpha$ and $r_t = \alpha z k_t^{\alpha-1}$.
5. A capital market and a labor market clear in each period: For all t , $K_{t+1} = N_{t-1} s_t - N_t \phi n_t$ and $L_{t+1} = N_t$.

Suppose that a sequence, $(x_t)_{t=0}^\infty$, is governed by a law of motion, $x_{t+1} = g(x_t)$, where $g : X \rightarrow X$. A point $\bar{x} \in X$ is a *stationary point* if $\bar{x} = g(\bar{x})$. A stationary point $\bar{x} \in X$ is *locally stable* if there exists a subset $\hat{X} \subset X$ such that for all $x_0 \in \hat{X}$ and $(x_t)_{t=0}^\infty$ satisfying $x_{t+1} = g(x_t)$, $\lim_{t \rightarrow \infty} x_t = \bar{x}$. A *steady-state equilibrium* is an equilibrium that is time-invariant.

3 Equilibrium analysis

Letting λ_1 and λ_2 be the Lagrange multipliers of the borrowing constraint (Equation (1)) and the lifetime budget constraint (Equation (4)), set the Lagrangian of a young agent's problem in period t :

$$L = U(n_t, c_{t+1}, d_{t+2}) + \lambda_1[\mu w_{t+1} - r_{t+1}\phi n_t] + \lambda_2 \left[w_{t+1} - c_{t+1} - r_{t+1}\phi n_t - \frac{d_{t+2}}{r_{t+2}} \right].$$

From the first-order conditions, I obtain

$$\frac{\beta}{c_{t+1}} = \lambda_2, \quad (6)$$

$$d_{t+2} = \frac{\gamma}{\beta} r_{t+2} c_{t+1}, \quad (7)$$

$$\frac{1}{n_t} = [\lambda_1 + \lambda_2] r_{t+1} \phi, \quad (8)$$

$$\lambda_1[\mu w_{t+1} - r_{t+1}\phi n_t] = 0. \quad (9)$$

3.1 Borrowing constraint does not bind

First, I investigate the case in which the borrowing constraint does not bind. That is, $\lambda_1 = 0$. This corresponds to a case in which the capital market is perfect.

From Equations (6) and (8), I have

$$r_{t+1}\phi n_t = \frac{c_{t+1}}{\beta}. \quad (10)$$

Applying Equations (7) and (10) to the lifetime budget constraint, I get

$$c_{t+1} = \frac{\beta w_{t+1}}{1 + \beta + \gamma}.$$

From this, $n_t = \frac{c_{t+1}}{\beta \phi r_{t+1}} = \frac{w_{t+1}}{\phi(1+\beta+\gamma)r_{t+1}}$. In equilibrium,

$$n_t = \frac{1-\alpha}{\phi(1+\beta+\gamma)\alpha} k_{t+1}. \quad (11)$$

Since $s_{t+1} = \frac{d_{t+2}}{r_{t+2}}$,

$$s_{t+1} = \frac{\gamma w_{t+1}}{1+\beta+\gamma} = \frac{\gamma(1-\alpha)z}{1+\beta+\gamma} k_{t+1}^\alpha. \quad (12)$$

The market-clearing condition based on Equations (11) and (12) implies

$$\begin{aligned} k_{t+1} &= \frac{K_{t+1}}{L_{t+1}} = \frac{s_t}{n_{t-1}} - \phi n_t = \frac{\gamma(1-\alpha)z k_t^\alpha}{1+\beta+\gamma} \frac{\phi(1+\beta+\gamma)\alpha}{(1-\alpha)k_t} - \phi \frac{1-\alpha}{\phi(1+\beta+\gamma)\alpha} k_{t+1} \\ &= \gamma \phi z \alpha k_t^{\alpha-1} - \frac{1-\alpha}{\alpha(1+\beta+\gamma)} k_{t+1}. \end{aligned}$$

Thus, the law of motion of capital per unit of labor is

$$k_{t+1} = \frac{\alpha^2(1+\beta+\gamma)\gamma\phi z}{\alpha(1+\beta+\gamma)+1-\alpha} k_t^{\alpha-1}.$$

Proposition 3.1. *There exists a unique steady-state equilibrium characterized by*

$$\bar{k}^{nb} := \left[\frac{\alpha^2(1+\beta+\gamma)\gamma\phi z}{\alpha(1+\beta+\gamma)+1-\alpha} \right]^{\frac{1}{2-\alpha}},$$

where the superscript of \bar{k} stands for "not binding." Moreover, \bar{k}^{nb} is locally stable. In a steady-state equilibrium, the fertility rate is

$$\bar{n}^{nb} := \frac{1-\alpha}{\phi(1+\beta+\gamma)\alpha} \bar{k}^{nb}. \quad (13)$$

Proof. See the appendix.

Q.E.D.

3.2 Borrowing constraint binds

Next, consider a case where the borrowing constraint binds, that is, $\lambda_1 > 0$. Since the borrowing constraint binds, from Equation (9), the fertility rate is

$$n_t = \frac{\mu}{\phi} \frac{w_{t+1}}{r_{t+1}} = \frac{\mu(1-\alpha)}{\phi\alpha} k_{t+1}. \quad (14)$$

The condition under which the borrowing constraint binds is derived from Equations (11) and (14). The condition is

$$\mu \leq \bar{\mu} := \frac{1}{1+\beta+\gamma} < 1.$$

Applying Equations (7) and (14) to the lifetime budget constraint, I obtain

$$c_{t+1} = \frac{(1-\mu)\beta}{\beta+\gamma} w_{t+1}.$$

Since $s_{t+1} = \frac{d_{t+2}}{r_{t+2}} = \frac{\gamma}{\beta} c_{t+1}$,

$$s_{t+1} = \frac{\gamma(1-\mu)}{\beta+\gamma} w_{t+1} = \frac{\gamma(1-\mu)(1-\alpha)z}{\beta+\gamma} k_{t+1}^\alpha. \quad (15)$$

The market-clearing condition based on Equations (14) and (15) implies that

$$\begin{aligned} k_{t+1} &= \frac{s_t}{n_{t-1}} - \phi n_t = \frac{\gamma(1-\mu)(1-\alpha)z k_t^\alpha}{\beta+\gamma} \frac{\phi\alpha}{\mu(1-\alpha)k_t} - \frac{\mu(1-\alpha)}{\alpha} k_{t+1} \\ &= \frac{\gamma(1-\mu)\phi}{(\beta+\gamma)\mu\alpha z} k_t^{\alpha-1} - \frac{\mu(1-\alpha)}{\alpha} k_{t+1}. \end{aligned}$$

From this, the law of motion of capital per unit of labor is

$$k_{t+1} = \frac{\alpha^2 \gamma (1-\mu) \phi z}{(\beta+\gamma)\mu[\alpha+\mu(1-\alpha)]} k_t^{\alpha-1}.$$

Proposition 3.2. *There exists a unique steady-state equilibrium characterized by*

$$\bar{k}^b := \left[\frac{\alpha^2 \gamma (1 - \mu) \phi z}{(\beta + \gamma) \mu [\alpha + \mu (1 - \alpha)]} \right]^{\frac{1}{2 - \alpha}}, \quad (16)$$

where the superscript of \bar{k} stands for “binding.” Moreover, \bar{k}^b is locally stable. In a steady-state equilibrium, the fertility rate is

$$\bar{n}^b := \frac{\mu(1 - \alpha)}{\phi \alpha} \bar{k}^b. \quad (17)$$

Proof. The proof is exactly the same as that of Proposition 3.1.

Q.E.D.

4 Comparative statics

In this section, I focus on a unique steady-state equilibrium characterized in the previous section, and investigate how the borrowing constraint affects the fertility rate and welfare.

4.1 Fertility rate

First, consider the effect of the borrowing constraint, μ , on the fertility rate in a steady-state equilibrium.

Proposition 4.1. *There exists a unique $\hat{\mu} \in (0, 1)$ such that*

$$\frac{d\bar{n}^b}{d\mu}(\mu) \begin{cases} > 0 & \text{if } \mu < \hat{\mu} \\ = 0 & \text{if } \mu = \hat{\mu} \\ < 0 & \text{if } \mu > \hat{\mu} \end{cases}.$$

This proposition states that the fertility rate increases as μ increases when μ is small, while it decreases as μ increases when μ is not small. The intuition behind this result is as follows. If μ increases, then the young agent can borrow more than before. Therefore, the young agent will have more children than before. This effect is expressed by the first μ in the RHS of Equation (17). If the agent borrows more, then

the capital stock will decrease. At the same time, a larger population means a larger labor force. These lead to a decrease in capital per unit of labor. Since prices are determined by capital per unit of labor in equilibrium, the interest rate increases and wage decreases, reducing the young agent's incentive to have children. In the RHS of Equation (17), \bar{k}^b expresses this effect.⁶ Thus, the stronger effect determines the population of agents.

Note that in this growth model, the GDP growth rate in a steady-state equilibrium, $\frac{Y_{t+1}}{Y_t}$, is equal to the population growth rate. Hence, the GDP growth rate also depends on μ , increasing or decreasing as μ increases, depending on whether μ is small or large.

The relationship between $\bar{\mu}$ and $\hat{\mu}$ shows how the fertility rate in a steady-state equilibrium changes as μ changes. When $\bar{\mu} \leq \hat{\mu}$, the fertility rate increases as μ increases until μ reaches $\bar{\mu}$. After that, the fertility rate stays constant at \bar{n}^{nb} . A more interesting case is when $\bar{\mu} > \hat{\mu}$. In this case, the fertility rate increases until μ reaches $\hat{\mu}$. After that, the fertility rate starts decreasing as μ increases, and stops when μ reaches $\bar{\mu}$. When μ is larger than $\bar{\mu}$, the fertility rate stays constant at \bar{n}^{nb} .

When does $\bar{\mu} > \hat{\mu}$ hold? Under a plausible assumption on β and γ , it can be shown that $\bar{\mu} > \hat{\mu}$.

Lemma 4.1. *Suppose $\beta \in (0, 1]$ and $\gamma \in (0, 1]$. Then, $\bar{\mu} > \hat{\mu}$.*

Proof. See the appendix.

Q.E.D.

Combining Proposition 4.1 and Lemma 4.1, the fertility rate is hump-shaped when μ is not too large and constant, at \bar{n}^{nb} , when μ is large. Figure 1 illustrates how the fertility rate in a steady-state equilibrium changes in μ .⁷

How does the real data look like? Each point on Figure 2 shows the data of household credit to GDP and the fertility rate for some country. Data of household credit to GDP, which is used as a proxy for the tightness of the borrowing constraint, are from Beck et al. (2012). The fertility rate is as of 2013. Figure 2 focuses on OECD countries, for which household credit to GDP data are available in Beck et al. (2012). The correlation coefficient is 0.106. Figure 2 does not show a clear hump-shaped relationship

⁶This effect might be more evident from Equation (14).

⁷The parameters in Figure 1 are $\beta = (.98)^{20}$, $\gamma = (.98)^{20} \times \beta$, $z = 1$, $\alpha = .35$ and $\phi = .03$. With these values, $\bar{\mu} = .4732$.

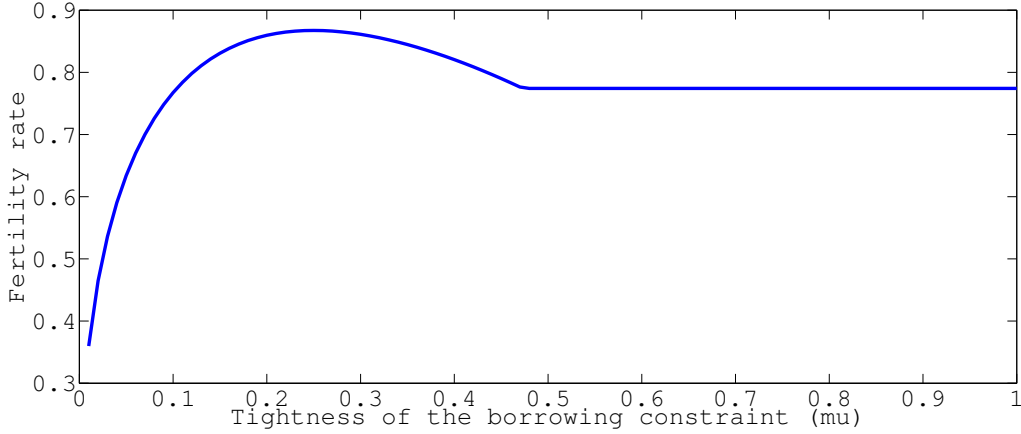


Figure 1: Steady-state fertility rate for different μ

between these two variables, as Proposition 4.1 suggests, but shows an ambiguous relationship between the borrowing constraint and the fertility rate, which contradicts the finding in Filoso and Papagni (2015). This result and Proposition 4.1 suggest that a general equilibrium effect may explain this ambiguous relationship between the borrowing constraint and the fertility rate.

4.2 Welfare

Next, I investigate how the borrowing constraint, μ , affects welfare. If the borrowing constraint does not bind, μ does not change welfare. Thus, this section focuses on a steady-state equilibrium where the borrowing constraint binds.⁸ In this paper, welfare is defined by the representative agent's lifetime utility in a steady-state equilibrium. Since agents are identical, the representative agent's lifetime utility can be considered as the average lifetime utility of an agent.⁹ Let (n, c, d) be a steady-state equilibrium allocation, given some μ . Welfare is defined by

$$W(\mu) := \ln(n) + \beta \ln(c) + \gamma \ln(d).$$

⁸From here on, I omit the superscript nb.

⁹This criterion is used in the literature. See, for instance, Cremer et al. (2006) and Miyazaki (2013).

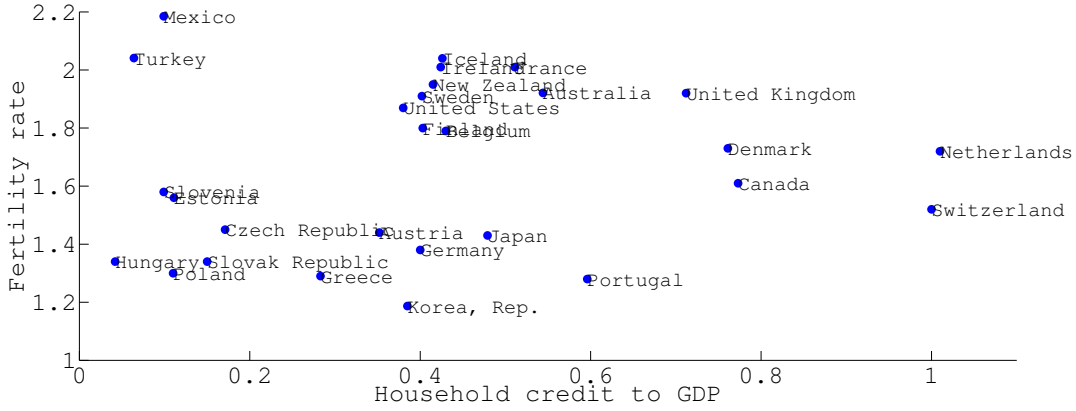


Figure 2: Fertility rate and Household credit to GDP in OECD countries

Proposition 4.2. *If $\alpha < \frac{1+\gamma}{1+\beta+2\gamma}$, then there exists a unique $\tilde{\mu} \in (0, 1)$ such that $\frac{dW}{d\mu}(\mu) > 0$ for $\mu < \tilde{\mu}$, $\frac{dW}{d\mu}(\mu) = 0$ for $\mu = \tilde{\mu}$, and $\frac{dW}{d\mu}(\mu) < 0$ for $\mu > \tilde{\mu}$. If $\alpha \geq \frac{1+\gamma}{1+\beta+2\gamma}$, then for all $\mu \in (0, 1)$, $\frac{dW}{d\mu}(\mu) < 0$.*

Proof. See the appendix.

Q.E.D.

The effect of μ on welfare depends on the size of α . When α is small enough, labor is more important for production than capital stock. In addition, having children provides utility to an agent. Thus, welfare is hump-shaped in μ . To the contrary, when α is large, capital stock is more important for production. Since borrowing in young age to have a child reduces capital stock in the next period, output will decrease. Thus, a large μ reduces welfare.

Figures 3 and 4 display welfare for each value of μ under the same parameter values in Figure 1, except for α . Note that I use $\alpha = 0.35$ for Figure 3, and $\alpha = 0.6$ for Figure 4. For both Figures, $\frac{1+\gamma}{1+\beta+2\gamma} = 0.5649$.

5 Conclusions

This paper studies the relationship between the exogenous borrowing constraint and the fertility rate in an OLG model for four-period-lived agents. The main finding is that even if the borrowing constraint is

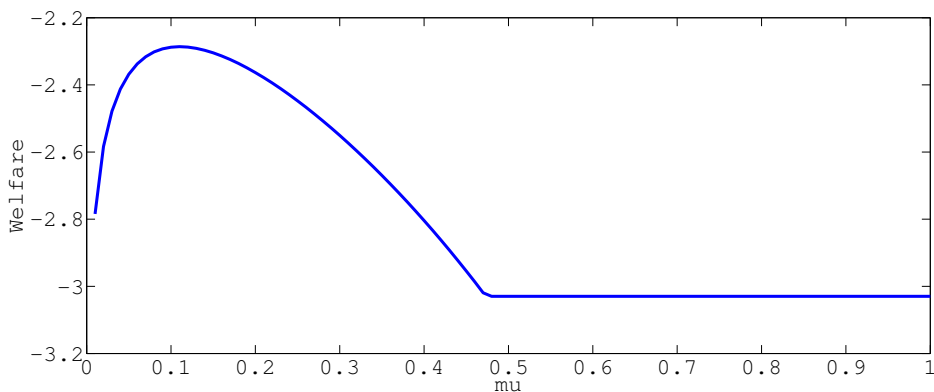


Figure 3: Welfare when $\alpha = 0.35$

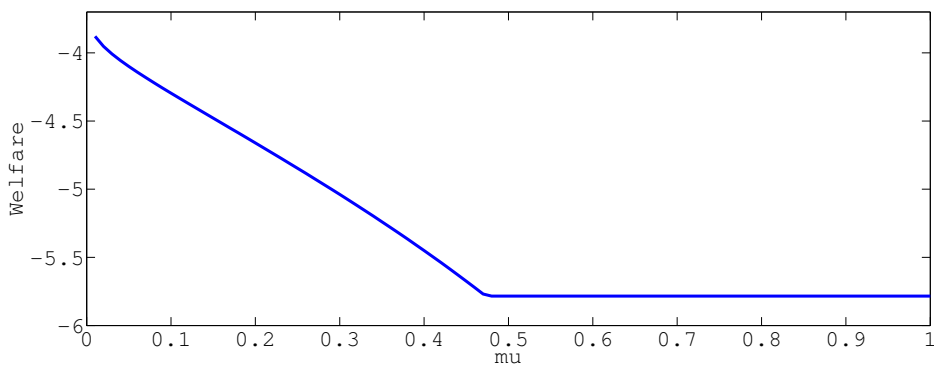


Figure 4: Welfare when $\alpha = 0.6$

relaxed, the fertility rate does not necessarily increase, as found in real data as well.

One possible policy implication is that to increase the fertility rate, borrowing for raising children should be constrained. At first sight, this might be counter-intuitive. Of course, if the borrowing constraint is relaxed so that the agent can borrow more to have children, the fertility rate will increase. In addition to this positive effect on the fertility rate, there exists a negative effect as well, which is often ignored in a partial equilibrium setting. If a young agent borrows more and has more children, the physical capital stock decreases and labor increases, increasing the debt burden. This will reduce a young agent's incentive to have many children. Thus, considering these two effects together, a borrowing constraint might benefit

an economy as it increases the fertility rate.

A Proofs

A.1 Proof of Proposition 3.1

Proof. The steady-state equilibrium capital per unit of labor is a solution to

$$k = \frac{\alpha^2(1 + \beta + \gamma)\gamma\phi z}{\alpha(1 + \beta + \gamma) + 1 - \alpha} k^{\alpha-1}.$$

The function, $g(k) := k$, is strictly increasing and continuous in k , and satisfies $g(0) = 0$ and $\lim_{k \rightarrow \infty} g(k) = \infty$.

The function, $h(k) := \frac{\alpha^2(1 + \beta + \gamma)\gamma\phi z}{\alpha(1 + \beta + \gamma) + 1 - \alpha} k^{\alpha-1}$, is continuous and strictly decreasing, and satisfies $\lim_{k \rightarrow 0} h(k) = \infty$, and $\lim_{k \rightarrow \infty} h(k) = 0$. Therefore, \bar{k}^{nb} is a unique solution to the above problem.

Taking the derivative of h with respect to k and evaluating it at \bar{k}^{nb} , I obtain $h'(\bar{k}^{nb}) = -1 + \alpha \in (-1, 0)$. Since $|h'(\bar{k}^{nb})| < 1$, \bar{k}^{nb} is locally stable (for instance, see Theorem 6.5 in Stokey et al. (1989)).

Plugging \bar{k}^{nb} into Equation (11) gives Equation (13).

Q.E.D.

A.2 Proof of Proposition 4.1

Proof. First, \bar{n}^b is rearranged to

$$\bar{n}^b = \frac{1 - \alpha}{\phi \alpha} \left[\frac{\alpha^2 \gamma \phi z}{\beta + \gamma} \right]^{\frac{1}{2-\alpha}} \mu^{\frac{1-\alpha}{2-\alpha}} \left[\frac{1 - \mu}{\alpha + \mu(1 - \alpha)} \right]^{\frac{1}{2-\alpha}}.$$

Let

$$M(\mu) := \mu^{\frac{1-\alpha}{2-\alpha}} \left[\frac{1 - \mu}{\alpha + \mu(1 - \alpha)} \right]^{\frac{1}{2-\alpha}}.$$

\bar{n}^b changes in line with μ just as $M(\mu)$ changes in accordance with μ . Hence, taking the derivative of $M(\mu)$ with respect to μ , I obtain

$$\begin{aligned}
\frac{dM(\mu)}{d\mu} &= \frac{1-\alpha}{2-\alpha} \mu^{-\frac{1}{2-\alpha}} \left[\frac{1-\mu}{\alpha+\mu(1-\alpha)} \right]^{\frac{1}{2-\alpha}} \\
&\quad + \frac{\mu^{\frac{1-\alpha}{2-\alpha}}}{2-\alpha} \left[\frac{1-\mu}{\alpha+\mu(1-\alpha)} \right]^{\frac{\alpha-1}{2-\alpha}} \left\{ -\frac{1}{\alpha+\mu(1-\alpha)} - \frac{(1-\mu)(1-\alpha)}{[\alpha+\mu(1-\alpha)]^2} \right\} \\
&= \frac{\mu^{-\frac{1}{2-\alpha}}}{2-\alpha} \left[\frac{1-\mu}{\alpha+\mu(1-\alpha)} \right]^{\frac{\alpha-1}{2-\alpha}} \left\{ \frac{(1-\alpha)(1-\mu)}{\alpha+\mu(1-\alpha)} - \frac{\mu[\alpha+\mu(1-\alpha)] + \mu(1-\mu)(1-\alpha)}{[\alpha+\mu(1-\alpha)]^2} \right\} \\
&= \frac{\mu^{-\frac{1}{2-\alpha}}}{2-\alpha} \left[\frac{1-\mu}{\alpha+\mu(1-\alpha)} \right]^{\frac{\alpha-1}{2-\alpha}} \frac{1}{[\alpha+\mu(1-\alpha)]^2} \\
&\quad \times \{ [\alpha+\mu(1-\alpha)](1-\alpha)(1-\mu) - \mu[\alpha+\mu(1-\alpha)] - \mu(1-\mu)(1-\alpha) \} \\
&= \frac{\mu^{-\frac{1}{2-\alpha}}}{2-\alpha} \left[\frac{1-\mu}{\alpha+\mu(1-\alpha)} \right]^{\frac{\alpha-1}{2-\alpha}} \frac{1}{[\alpha+\mu(1-\alpha)]^2} \\
&\quad \times \{ -(1-\alpha)^2\mu^2 - \alpha[2(1-\alpha)+1]\mu + (1-\alpha)\alpha \}.
\end{aligned}$$

Let $m(\mu) := -(1-\alpha)^2\mu^2 - \alpha[2(1-\alpha)+1]\mu + (1-\alpha)\alpha$. The function $m(\mu)$ is hump-shaped and continuous, and satisfies $m(0) = (1-\alpha)\alpha > 0$ and $m(1) = -1$. This completes the proof. *Q.E.D.*

A.3 Proof of Lemma 4.1

Proof. If $\beta \in (0, 1]$ and $\gamma \in (0, 1]$, then $\bar{\mu} = \frac{1}{1+\beta+\gamma} \geq \frac{1}{3}$. Plugging $\frac{1}{3}$ into $m(\mu)$, I have

$$\begin{aligned}
m(1/3) &= -\frac{(1-\alpha)^2}{9} - \frac{\alpha[2(1-\alpha)+1]}{3} + (1-\alpha)\alpha = -\frac{4}{9}\alpha^2 + \frac{2}{9}\alpha - \frac{1}{9} \\
&= -\frac{1}{9} \left[4 \left(\alpha - \frac{1}{4} \right)^2 + \frac{3}{4} \right] < 0
\end{aligned}$$

for all $\alpha \in (0, 1)$. From the shape of $m(\mu)$, $\frac{1}{3} > \hat{\mu}$. Therefore, $\bar{\mu} > \hat{\mu}$ holds. *Q.E.D.*

A.4 Proof of Proposition 4.2

Proof. Since in equilibrium $d = \frac{\gamma}{\beta}rc$, $W(\mu) = \ln(n) + (\beta + \gamma)\ln(c) + \gamma\ln(r) + \gamma\ln(\gamma/\beta)$. Taking the derivative of W with respect to μ , I obtain

$$\frac{dW}{d\mu}(\mu) = \frac{1}{n} \frac{dn}{d\mu} + \frac{\beta + \gamma}{c} \frac{dc}{d\mu} + \frac{\gamma}{r} \frac{dr}{d\mu}.$$

From Equation (17),

$$\frac{1}{n} \frac{dn}{d\mu} = \frac{\phi\alpha}{k\mu(1-\alpha)} \left[\frac{1-\alpha}{\phi\alpha} k + \frac{\mu(1-\alpha)}{\phi\alpha} \frac{dk}{d\mu} \right] = \frac{1}{\mu} + \frac{1}{k} \frac{dk}{d\mu}. \quad (18)$$

From Equation (15),

$$\begin{aligned} \frac{\beta + \gamma}{c} \frac{dc}{d\mu} &= (\beta + \gamma) \frac{\beta + \gamma}{(1-\mu)\beta z(1-\alpha)k^\alpha} \left[-\frac{\beta}{\beta + \gamma} z(1-\alpha)k^\alpha + \frac{(1-\mu)\beta}{\beta + \gamma} z\alpha(1-\alpha)k^{\alpha-1} \frac{dk}{d\mu} \right] \\ &= -\frac{\beta + \gamma}{1-\mu} + \frac{(\beta + \gamma)\alpha}{k} \frac{dk}{d\mu}. \end{aligned} \quad (19)$$

Since $r = z\alpha k^{\alpha-1}$,

$$\frac{\gamma}{r} \frac{dr}{d\mu} = \frac{\gamma}{z\alpha k^{\alpha-1}} z\alpha(\alpha-1)k^{\alpha-2} \frac{dk}{d\mu} = \frac{\gamma(\alpha-1)}{k} \frac{dk}{d\mu}. \quad (20)$$

From Equation (16),

$$\begin{aligned} \frac{dk}{d\mu} &= \frac{1}{2-\alpha} \left[\frac{\alpha^2\gamma(1-\mu)\phi z}{(\beta + \gamma)\mu[\alpha + \mu(1-\alpha)]} \right]^{\frac{1}{2-\alpha}-1} \\ &\quad \times \left\{ \frac{-\alpha^2\gamma\phi z}{(\beta + \gamma)\mu[\alpha + \mu(1-\alpha)]} - \frac{\alpha^2\gamma(1-\mu)\phi z}{(\beta + \gamma)\mu^2[\alpha + \mu(1-\alpha)]^2} \{\alpha + 2\mu(1-\alpha)\} \right\} \\ &= \frac{1}{2-\alpha} k \left\{ -\frac{1}{1-\mu} - \frac{\alpha + 2\mu(1-\alpha)}{\mu[\alpha + \mu(1-\alpha)]} \right\}. \end{aligned} \quad (21)$$

By using Equations (18), (19), (20) and (21),

$$\frac{dW}{d\mu}(\mu) = \frac{1}{(2-\alpha)\mu(1-\mu)[\alpha+\mu(1-\alpha)]} \times \left\{ \begin{array}{l} (2-\alpha)(1-\mu)[\alpha+\mu(1-\alpha)] - (\beta+\gamma)(2-\alpha)[\alpha+\mu(1-\alpha)] \\ -[1+(\beta+\gamma)\alpha-\gamma(1-\alpha)]\mu[\alpha+\mu(1-\alpha)] \\ -[1+(\beta+\gamma)\alpha-\gamma(1-\alpha)](1-\mu)[\alpha+2\mu(1-\alpha)] \end{array} \right\}.$$

Letting

$$\begin{aligned} G(\mu) &:= (2-\alpha)(1-\mu)[\alpha+\mu(1-\alpha)] - (\beta+\gamma)(2-\alpha)[\alpha+\mu(1-\alpha)] \\ &\quad - [1+(\beta+\gamma)\alpha-\gamma(1-\alpha)]\mu[\alpha+\mu(1-\alpha)] \\ &\quad - [1+(\beta+\gamma)\alpha-\gamma(1-\alpha)](1-\mu)[\alpha+2\mu(1-\alpha)] \\ &= -(1+2\beta+3\gamma)(1-\alpha)^2\mu^2 + \{(2+3\beta+5\gamma)\alpha^2 - (3+4\beta+8\gamma)\alpha + 2\gamma\}\mu \\ &\quad + \alpha\{-(1+\beta+2\gamma)\alpha + 1 + \gamma\}, \end{aligned}$$

then $\frac{dW}{d\mu}(\mu) = \frac{G(\mu)}{(2-\alpha)\mu(1-\mu)[\alpha+\mu(1-\alpha)]}$.

Note that G is hump-shaped, $G(1) = -1 - 2\beta - \gamma(1 + \alpha) < 0$, and $G(0) = \alpha\{-(1 + \beta + 2\gamma)\alpha + 1 + \gamma\}$. Thus, $G(0) > 0$ if and only if $\alpha < \frac{1+\gamma}{1+\beta+2\gamma}$. If $G(0) > 0$, then there exists $\tilde{\mu}$ such that $\frac{dW}{d\mu} > 0$ if and only if $\mu < \tilde{\mu}$.¹⁰ When $\alpha \geq \frac{1+\gamma}{1+\beta+2\gamma}$, how $G(\mu)$ looks like depends on the coefficient of μ in $G(\mu)$. Thus, let $J(\alpha) := (2 + 3\beta + 5\gamma)\alpha^2 - (3 + 4\beta + 8\gamma)\alpha + 2\gamma$. Note that $J(0) = 2\gamma > 0$ and $J(1) = -1 - \beta - \gamma < 0$. Since J is U-shaped, if $J\left(\frac{1+\gamma}{1+\beta+2\gamma}\right) < 0$, then for all $\alpha \geq \frac{1+\gamma}{1+\beta+2\gamma}$, $J(\alpha) < 0$. Then,

$$J\left(\frac{1+\gamma}{1+\beta+2\gamma}\right) = -\frac{1+6\gamma+10\gamma^2+3\gamma^3+4\beta^2+5\beta\gamma^2+2\beta^2\gamma+13\beta\gamma}{(1+\beta+2\gamma)^2} < 0.$$

From this, if $\alpha \geq \frac{1+\gamma}{1+\beta+2\gamma}$, then $G(\mu)$ takes the maximum at some $\mu < 0$, $G(0) < 0$ and $G(1) < 0$. This implies that $\frac{dW}{d\mu}(\mu) < 0$ for all $\mu \in (0, 1)$. *Q.E.D.*

¹⁰ $\tilde{\mu}$ is a solution to $G(\mu) = 0$.

References

- Robert Barro and Gary Becker. Fertility choice in an endogenous growth model. *Econometrica*, 57(2): 481–501, 1989.
- Thorsten Beck, Berrak Büyükkarabacak, Felix K Rioja, and Neven T Valev. Who gets the credit? and does it matter? household vs. firm lending across countries. *The BE Journal of Macroeconomics*, 12 (1), 2012.
- Gary Becker and Robert Barro. A reformulation of the economic theory of fertility. *The Quarterly Journal of Economics*, 103(1):1–25, 1988.
- Helmuth Cremer, Firouz Gahvari, and Pierre Pestieau. Pensions with endogenous and stochastic fertility. *Journal of Public Economics*, 90(12):2303–2321, 2006.
- Z. Eckstein and K. Wolpin. Endogenous fertility and optimal population size. *Journal of Public Economics*, 27(1):93–106, 1985.
- Valerio Filoso and Erasmo Papagni. Fertility choice and financial development. *European Journal of Political Economy*, 37:160–177, 2015.
- Tullio Jappelli and Marco Pagano. Saving, growth, and liquidity constraints. *The Quarterly Journal of Economics*, 109(1):83–109, 1994.
- Koichi Miyazaki. Pay-as-you-go social security and endogenous fertility in a neoclassical growth model. *Journal of Population Economics*, 26(3):1233–1250, 2013.
- Erasmo Papagni. Household borrowing constraints, fertility dynamics, and economic growth. *Journal of Economic Dynamics and Control*, 30(1):27–54, 2006.
- Nancy L. Stokey, Robert E. Lucas Jr., and Edward C. Prescott. *Recursive methods in economic dynamics*. Harvard University Press, 1989.

Berthold U. Wigger. Pay-as-you-go financed public pensions in a model of endogenous growth and fertility. *Journal of Population Economics*, 12(4):625–640, 1999.

Junsen Zhang and Junxi Zhang. Social security, intergenerational transfers, and endogenous growth. *Canadian Journal of Economics*, 31(5):1225–1241, 1998.